Introduction to minimal and maximal surfaces

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Overview

Brief History

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- Soap Experiment:

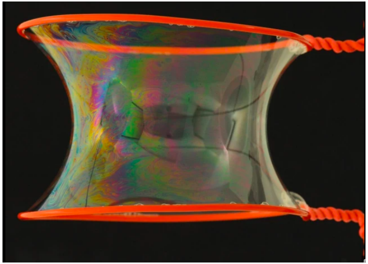

Figure: Soap film and minimal surface

Results about 2-D manifolds [Osserman, 1997]

Let M be a 2-dimensional, oriented surface with a Riemannian metric, and let ∇ denote its Riemannian connection. For a smooth function $f : M \to \mathbb{R}$, we will denote its differential by df and its gradient by grad f such that for any $V \in TM$, $df(V) =< \text{grad} f, V >$. And, Laplacian of f is defined as $\Delta f := div(gradf)$.

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Moreover, with the given metric and orientation this $J := \text{Rot}_{\pi/2}$ is unique and satisfies:

$$g(Ju, Jv) = g(u, v), \quad g(u, Ju) = 0, ||Ju|| = ||u|| \text{ and } \omega(u, v) = g(Ju, v).$$

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For a smooth harmonic function $f : M \to \mathbb{R}$, locally we can define its harmonic conjugate say f^* such that $\mathrm{grad} f^* = \mathrm{Rot}_{\pi/2}(\mathrm{grad} f)$. Moreover, $df^* = -df \circ \mathrm{Rot}_{\pi/2}$, and the locally defined mapping given by (f, f^*) is conformal wherever $\mathrm{grad} f \neq 0$. Thus, $f + \iota f^*$ is a local conformal coordinate in a neighborhood of any point $p \in M$ where $df_p \neq 0$.

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Suppose our surface M is given as an oriented immersed surface in $\mathbb{R}^3$. Let X denote its position vector; we think of X as an immersion $X : M \to \mathbb{R}^3$. Let M inherits a Riemannian metric (.) from $\mathbb{R}^3$. Let $N : M \to S^2$ be its Gauss map and $H : M \to \mathbb{R}$ be the mean curvature map corresponding the choice on N.
An important formula with strong consequences for minimal surfaces is: $\Delta X = 2HN$.

Equivalent Definitions of Minimal Surfaces

Harmonic Coordinate Functions

Let $X = (X_1, X_2, X_3) : M \to \mathbb{R}^3$ be a conformal immersion of an oriented Riemannian surface into $\mathbb{R}^3$. Then, X is called a minimal immersion iff each X_i is harmonic on M, i.e.,

$$\Delta X_i = 0 \quad \text{for } i = 1, 2, 3,$$

where Δ is the Riemannian Laplacian on M. The image $X(M)$ is called a minimal surface.

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Mean Curvature Vanishes

A surface $M \subset \mathbb{R}^3$ is minimal if and only if its mean curvature H vanishes identically: $H \equiv 0$.

Continued...

Let $\Omega \subset M$ be an orientable subdomain with compact closure. Perturb the inclusion X on Ω normally by a compactly supported smooth function $u \in C_c^\infty(\Omega)$. For each t, $A(t)$ is the area of the perturbed surface $X_t(\Omega)$. The area is given by:

$$A'(0) = -2 \int_\Omega uHdA,$$

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A surface $M \subseteq \mathbb{R}^3$ is minimal if and only if it is a critical point of the area functional for all compactly supported variations.

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$$ds^2 = \frac{1}{4}(|g(z)| + |g(z)|^{-1})^2 |f(z)|^2.$$

Related

Branched minimal surfaces

The points in M where X fails to be an immersion or metric degernates are called branched points and the minimal surface $X(M)$ is called branched/generalised minimal surface.

Associate surfaces

The associate surfaces to X are the minimal immersions X_θ given by the Weierstrass data $\{g, e^{i\theta}dh\}$. When $\theta = \pi/2$, $X_{\pi/2}$ is called a conjugate/assoiciate surface with data $\{g, idh\}$.

Remark:The Weierstrass data $\{e^{i\theta}g, dh\}$ produces the same minimal surface as rotated by an angle θ around the vertical axis.

Examples of minimal surfaces

Catenoid

Let $M = \mathbb{C} - \{0\}$, $g = z$ and $dh = \frac{dz}{z}$. By the Divergence Theorem, it is sufficient to show that period vanishes on the first homology class in M.
Moreover, for a Riemann surface of genus g with r punctures, $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{2g+r-1}$ and the cardinality of generators of $H_1(M, \mathbb{Z}) = 2g + r - 1$.

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Helicoid

Let $M = \mathbb{C} - \{0\}$, $g = z$ and $dh = \frac{idz}{z}$. Let X^* be the minimal surface for the data, but observe that X^* is not well-define on whole M, it is multi-valued function.

Examples

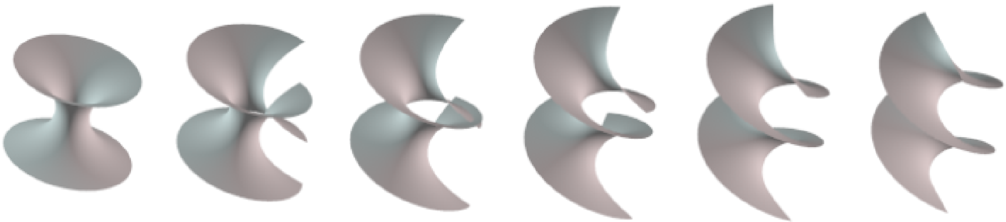

Figure: A conjugate family of minimal surfaces between the catenoid and the helicoid

Osserman's Theorem

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Let $X : M \to \mathbb{R}^3$ be a complete conformal minimal immersion with finite total curvature. Then

- M is conformally diffeomorphic to $\overline{M}_k - \{p_1, ..., p_r\}$ where $\overline{M}_k$ is a compact Riemann surface of genus k,
- X is proper,
- g and dh extends meromorphically on $\overline{M}_k$,
- the total curvature $\kappa = -4\pi \deg(g) \leq -4\pi(k + r - 1)$.

The Lorentz-Minkowski Space $\mathbb{E}^3_1$ [López, 2014]

The 3-dimensional Lorentz-Minkowski space, $\mathbb{E}^3_1$, is a vector space $\mathbb{R}^3$ equipped with a bilinear form $< -, - > : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by $< (u_1, u_2, u_3), (v_1, v_2, v_3) > := u_1 v_1 + u_2 v_2 - u_3 v_3$, where $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3$. In the literature, this is called the Lorentz metric.

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A vector $v \in \mathbb{E}_1^3$ is said to be spacelike if $\langle v, v \rangle > 0$ or $v = 0$, timelike if $\langle v, v \rangle < 0$, and lightlike if $\langle v, v \rangle = 0$ and $v \neq 0$.

Maxface [Umehara and Yamada, 2006]

Definition

Let $X : M \to \mathbb{E}^3_1$ be a smooth map on a 2-dimensional oriented manifold M, then (X, M) is called a generalised maximal immersion if

- the pullback metric $X^*(\langle ., . \rangle)$ on T_pM is positive definite for almost all p in M,
- mean curvature is zero almost everywhere,
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The singularity set of X is defined as

$$Sing(X) := \{p \in M : rank(dX|_p) < 2\}.$$

A generalized maximal immersion is called Maxface if its differential map has exactly rank 1 at all singular points.

Weierstrass-Enneper representation for the maximal map ([Kobayashi, 1983, Umehara and Yamada, 2006])

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Furthermore, the pullback metric on M is $ds^2 = \frac{1}{4}\left(|g|^{-1} - |g|\right)^2 |\omega|^2$.

Construction of maxface

The singular Björling problem which asks, whether there exists a spacelike maximal surface that contains a prescribed curve in $\mathbb{E}_1^3$ as a singular set and prescribed null directions on the curve as the normal directions to the surface. It's solution is given by Kim and Yang in [Kim and Yang, 2007].

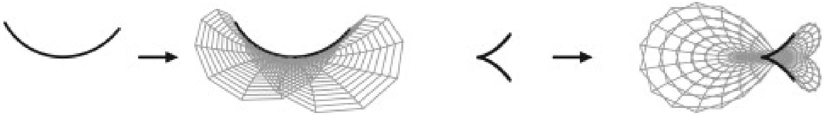

Figure: Generalised maximal surfaces which contains prescribed curve as singularities

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2. α be a null curve and β be a null vector field defined on I and for all $t \in I$, $\langle \alpha'(t), \beta(t) \rangle = 0$. Moreover, either $\alpha'(t) \neq 0$ or $\beta(t) \neq 0$ for all $t \in I$.
3. The map given by $\phi(\lambda(t)) = \alpha'(t) - i\beta(t)$ has an extension $\phi(z)$ on $I \subset \Omega \subset \mathbb{C}$.
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$$G(t) := \begin{cases} \frac{\alpha_1'(t) + i\alpha_2'(t)}{\alpha_3'(t)} & \text{if } \alpha'(t) \neq 0 \\ \frac{\beta_1(t) + i\beta_2(t)}{\beta_3(t)} & \text{if } \beta_3(t) \neq 0 \end{cases}$$

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and $|g(w)|$ is not identically equal to 1. For the data $\{\lambda, \alpha, \beta\}$, if $Re \int_{\gamma_j} \phi(z)dz$ vanishes for all loops γ_j on a neighborhood Ω of trace of λ. Then the map $X := (X_1, X_2, X_3) : \Omega \to \mathbb{E}_1^3$, given by

$$X_{\lambda,\alpha,\beta}(z) = Re \left(\int_{\lambda(t_0)}^{z} \phi(w)dw \right)$$

is well defined and it gives a generalized maximal immersions on Ω, containing the trace

Examples

1. **Lorentzian Catenoid:** Let $M = \mathbb{C} \setminus \{0\}$, $g(z) = z$ and $\omega(z) = \frac{dz}{z}$. Then $X : M \to \mathbb{E}_1^3$ determined by equation (2) is a maxface. The set of singularities is $\{|z| = 1\}$ which is compact and its image by X is the origin in $\mathbb{E}_1^3$. So, every point of singularity set exhibits conelike singularity.
2. **Lorentzian Helicoid:** For, $M = \mathbb{C} \setminus \{0\}, g = z$ and $\omega = \frac{i}{z} dz$ has folded singularity on each $|z| = 1$.
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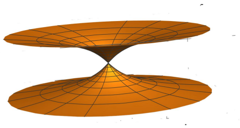
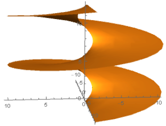
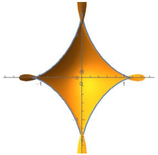


Lorentzian Catenoid



Lorentzian helicoid



Lorentz Enneper

Companion surfaces

Companion Surface [Umehara and Yamada, 2006]

If a maxface has Weierstrass data (M, g, ω), then the modified data (M, ig, idh) produce a conformal minimal immersion in $\mathbb{R}^3$, called the companion surface.

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Costa surface as a minimal immersion

Consider the Riemann surface

$$M = \left\{(z, w) \in \mathbb{C} \times \mathbb{C} \cup \{(\infty, \infty)\} \;\middle|\; w^2 = z(z^2 - 1)\right\} \setminus \{(0,0), (\pm 1, 0)\},$$

$$g(z) = \frac{a}{w}, \qquad dh = \frac{2a}{z^2 - 1}\, dz, \qquad a \in \mathbb{R}^+ \setminus \{0\},$$

give the Weierstrass data for Costa's minimal surface.

Costa Surface

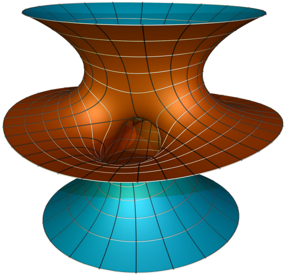

Figure: Costa minimal surface

Suppose there exists a maxface for which the companion is the Costa surface. Then its data should be

$$\{M, -ia/w, 2ia.dz/(z^2 - 1)\}.$$

Costa Surface

A period around $(-1, 0)$

Let γ be a one–sheeted loop around $(-1, 0)$ that does not contain $(1, 0)$. Then

$$\oint_{\gamma} \frac{2ia}{z^2 - 1}\, dz = \oint_{\gamma} \frac{2ia}{(z-1)(z+1)}\, dz = 2\pi a \neq 0.$$

Conclusion

The height differential dh has a non–trivial period around γ, hence the period problem is *not* solved for $a \neq 0$; extra global constraints are needed to obtain a well–defined Costa surface.

Cone-like, swallowtail singularities [Fujimori et al., 2009] [Fujimori et al., 2007]

Let $X : M \to \mathbb{E}^3_1$ be a maxface and $\sum_0$ be a set of singular points. Then a point $p \in \sum_0$ is called

Generalised cone-like singular point

if $\sum_0$ is connected, compact and image $X(\sum_0)$ is a single point. Moreover, if there is a neighborhood U of $\sum_0$ such that $X(U \setminus \sum_0)$ is embedded, then p is called cone-like singular point.

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Swallowtail

$Re(\alpha) \neq 0$, $Im(\alpha) = 0$ and for $\beta := \frac{g}{g'}\alpha'$, $Re(\beta) \neq 0$.

Global study of maxfaces

In the constuction of maxfaces various questions arise, such as their embeddedness, completeness, the types of ends they possess, their genus, and the nature of their singularities.

1. Lorentzian Catenoid is a genus-zero, complete, embedded maxface with two catenoidal ends.
2. In 2006 [Kim and Yang, 2006], Kim and Yang family of complete generalised maximal surfaces, M_g, of genus $g \geq 2$ with two ends which are not embedded and have $4(g+1)$ swallowtail singularities.
3. In 2023 [Kumar and Mohanty, 2023], give the existence of a genus-zero complete maximal map with a prescribed singularity set and an arbitrary number of simple and complete ends
4. In 2024 [Bardhan et al., 2024], give existence of new higher-genus maxfaces with Enneper end.

Configuration [Chen et al., 2024]

We want to construct a family of maxfaces that look like horizontal (spacelike) planes connected by small necks.

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- Consider L horizontal planes, labeled by integers $l \in [1, L]$,
- $n_l > 0$ necks at level l, that is, between the planes l and $l+1$, $1 \leq l < L$, denote $N = \sum n_l$ for the total number of necks,
- to each neck associate a complex number $p_{l,k} \in \mathbb{C}$ indicating its horizontal limit position at $t = 0$,
- to each plane is associated a real number Q_l, indicating the logarithmic growth of the catenoid ($Q_l \neq 0$) or planar ends ($Q_l = 0$).

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We write $p = (p_{l,k})_{1 \leq l < L, 1 \leq k \leq n_l}$ and $Q = (Q_l)_{1 \leq l < L}$. The pair (p, Q) is called a *configuration*. Given a configuration (p, Q), let c_l be the real numbers which corresponds to the "size" of the necks at level l.

Balanced and rigid configuration

For each neck (l, k) in a configuration, we define the force $F_{l,k}$ on the neck as

$$F_{l,k} := \sum_{1 \le i \ne k \le n_l} \frac{2c_l^2}{p_{l,k} - p_{l,i}} - \sum_{i=1}^{n_{l+1}} \frac{c_l c_{l+1}}{p_{l,k} - p_{l+1,i}} - \sum_{i=1}^{n_{l-1}} \frac{c_l c_{l-1}}{p_{l,k} - p_{l-1,i}}.$$

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In terms of c_l, define

$$W := \sum_{l=1}^{L} \sum_{k=1}^{n_l} p_{l,k} F_{l,k} = \sum_{l=1}^{L-1} n_l(n_l - 1)c_l^2 - \sum_{l=1}^{L-2} n_l n_{l+1} c_l c_{l+1} = 0.$$

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Definition 1

A configuration is *balanced* if $F_{l,k} = 0$ for all $1 \leq l < L$ and $1 \leq k \leq n_l$, and is *rigid* if the differential of $F = (F_{l,k})_{1\leq l<L, 1\leq k\leq n_l}$ with respect to p has a complex rank of $n-2$.

Main Theorem [Chen et al., 2024]

Theorem 2

Let (p, Q) be a balanced and rigid configuration such that the differential of $(c_1, ..., c_{L-1}) \mapsto W$ has rank 1. Then, for sufficiently small t, there is a smooth family M_t of complete maxfaces of genus $N - L + 1$ with the following asymptotic behaviors as $t \to 0$

- *the logarithmic growths of L space like ends converge to Q_l.*
- *After suitable scalings, the necks at level l converge to Lorentzian catenoids;*
- *M_t scaled by t converges to an L-sheeted space-like plane with singular points at $p_{l,k}$.*

Moreover, M_t is embedded in a wider sense for sufficiently small t if $Q_1 < Q_2 < \cdots < Q_L$.

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Thank You!!