Node-opening construction of maxfaces

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Overview

The Lorentz-Minkowski Space $\mathbb{E}^3_1$

The 3-dimensional Lorentz-Minkowski space, $\mathbb{E}^3_1$, is a vector space $\mathbb{R}^3$ equipped with a bilinear form $< -, - > : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by $< (u_1, u_2, u_3), (v_1, v_2, v_3) > := u_1 v_1 + u_2 v_2 - u_3 v_3$, where $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3$. In the literature, this is called the Lorentz metric.

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A vector $v \in \mathbb{E}_1^3$ is said to be spacelike if $\langle v, v \rangle > 0$ or $v = 0$, timelike if $\langle v, v \rangle < 0$, and lightlike if $\langle v, v \rangle = 0$ and $v \neq 0$.

Maxface [Umehara and Yamada, 2006]

Definition

Let $X : M \to \mathbb{E}_1^3$ be a smooth map on a 2-dimensional oriented manifold M, then (X, M) is called a generalised maximal immersion if

- the pullback metric $X^*(\langle ., . \rangle)$ on T_pM is positive definite for almost all p in M,
- mean curvature is zero almost everywhere,
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The singularity set of X is defined as

$$Sing(X) := \{p \in M : rank(dX|_p) < 2\}.$$

A generalized maximal immersion is called Maxface if its differential map has exactly rank 1 at all singular points. Maxfaces always have a non-isolated singularity set./compact?

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$$X : M \to \mathbb{E}^3_1, \;\; X(z) := \mathrm{Re} \int_{z_0}^{z} \Big(\frac{1}{2}(g^{-1} + g)\omega, \frac{i}{2}(g^{-1} - g)\omega, \omega \Big) \tag{1}$$

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Furthermore, the pullback metric on M is $ds^2 = \frac{1}{4} \left(|g|^{-1} - |g|\right)^2 |\omega|^2$.

Cone-like, swallowtail singularities [Fujimori et al., 2009] [Fujimori et al., 2007]

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Generalised cone-like singular point

if $\sum_0$ is connected, compact and image $X(\sum_0)$ is a single point. Moreover, if there is a neighborhood U of $\sum_0$ such that $X(U \setminus \sum_0)$ is embedded, then p is called cone-like singular point.

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Swallowtail

$Re(\alpha) \neq 0$, $Im(\alpha) = 0$ and for $\beta := \frac{g}{g'}\alpha'$, $Re(\beta) \neq 0$.

Global study of generalised maximal immersions or maxfaces

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1. Lorentzian Catenoid is a genus-zero, complete, embedded maxface with two catenoidal ends.
2. In [Kim and Yang, 2006], Kim and Yang gave a family of complete generalised maximal surfaces, M_g, of genus $g \geq 2$ with two ends which are not embedded and have $4(g+1)$ swallowtail singularities.
3. In [Bardhan et al., 2024], Bardhan, Biswas, and Kumar established the existence of new higher-genus maxfaces with Enneper end.
4. In [Chen et al., 2024], Chen, – , Kumar, and Mohanty showed the existence of a family of complete, embedded (in the wider sense) maxfaces with an arbitrary finite genus and an arbitrary number of spacelike ends (catenoidal and planar).

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We write $p = (p_{l,k})_{1 \leq l < L, 1 \leq k \leq n_l}$ and $Q = (Q_l)_{1 \leq l < L}$. The pair (p, Q) is called a *configuration*. Given a configuration (p, Q), let c_l be the real numbers which corresponds to the "size" of the necks at level l.

Balanced and rigid configuration

For each neck (l, k) in a configuration, we define the force $F_{l,k}$ on the neck as

$$F_{l,k} := \sum_{1 \leq i \neq k \leq n_l} \frac{2c_l^2}{p_{l,k} - p_{l,i}} - \sum_{i=1}^{n_{l+1}} \frac{c_l c_{l+1}}{p_{l,k} - p_{l+1,i}} - \sum_{i=1}^{n_{l-1}} \frac{c_l c_{l-1}}{p_{l,k} - p_{l-1,i}}.$$

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Definition

A configuration is *balanced* if $F_{l,k} = 0$ for all $1 \leq l < L$ and $1 \leq k \leq n_l$, and is *rigid* if the differential of $p = (p_{l,k}) \to F = (F_{l,k})_{1 \leq l < L, 1 \leq k \leq n_l}$ with respect to p has a complex rank of $N - 2$.

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A necessary condition fo the balance is:

$$W := \sum_{l=1}^{L}\sum_{k=1}^{n_l} p_{l,k}F_{l,k} = \sum_{l=1}^{L-1} n_l(n_l-1)c_l^2 - \sum_{l=1}^{L-2} n_l n_{l+1} c_l c_{l+1} = 0.$$

Main Theorem

Theorem (Hao,___,Kumar, Mohanty [Chen et al., 2024])

Let (p, Q) be a balanced and rigid configuration such that the differential of $(c_1, ..., c_{L-1}) \mapsto W$ has rank 1. Then, for sufficiently small t, there is a smooth family M_t of complete maxfaces of genus $N - L + 1$ with the following asymptotic behaviors as $t \to 0$

- the logarithmic growths of L space like ends converge to Q_l.
- After suitable scalings, the necks at level l converge to Lorentzian catenoids;
- M_t scaled by t converges to an L-sheeted space-like plane with singular points at $p_{l,k}$.

Moreover, M_t is embedded in a wider sense for sufficiently small t if $Q_1 < Q_2 < \cdots < Q_L$.

Example

The Costa–Hoffman–Meeks (CHM) configurations are given by

$$L = 3, \quad n_1 = 1, \quad n_2 = m,$$
$$p_{1,1} = 0, \quad p_{2,m} = e^{2k\pi i/m}, \quad 1 \leq k \leq m,$$
$$Q_1 = 1 - m, \quad Q_2 = -1, \quad Q_3 = m \quad (\text{so } c_1 = m - 1, \quad c_2 = 1).$$

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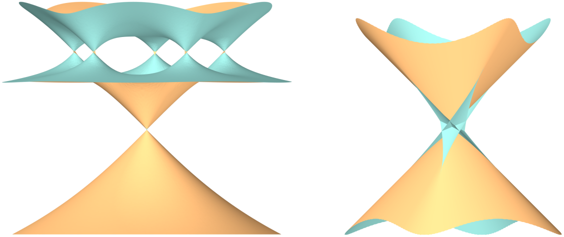

Figure: CHM surface for m=5, having cuspidal edges and swallowatails

Riemann surface construction

Node-opening construction (Martin Traizet [Traizet, 2002]): We construct the Riemann surface by node-opening as follows:
To each of the L horizontal (spacelike) planes is associated a copy of the complex plane $\mathbb{C}$, which can be seen as the Riemann sphere with puncture at ∞. The L copies of $\mathbb{C}$, as well as their punctures, are then indexed by l, $1 \leq l \leq L$.

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Choose an ε sufficiently small and independent of l and k. Use $v_{l,k} = 1/g_l$ and $w_{l+1,k} = 1/g_{l+1}$ as local complex coordinate in a neighbourhood of $a_{l,k}$ and $b_{l,k}$ respectively.

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$$g_l := \sum_{k=1}^{n_l} \frac{\alpha_{l,k}}{z - a_{l,k}} + \sum_{k=1}^{n_{l-1}} \frac{\beta_{l-1,k}}{z - b_{l-1,k}}.$$

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For $0 < t < \varepsilon$, we may remove the disks $|v_{I,k}| \leq t^2/\varepsilon$ and $|w_{I,k}| \leq t^2/\varepsilon$, and identify the annuli obtained

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Define a meromorphic function g on the disjoint union $\mathbb{C}_1 \cup \ldots \cup \mathbb{C}_L$ by

$$g(z) = \begin{cases} tg_l(z) & \text{if } z \in \mathbb{C}_l \text{ and } l \text{ is odd,} \\ 1/(tg_l(z)) & \text{if } z \in \mathbb{C}_l \text{ and } l \text{ is even.} \end{cases}$$

Let $\tilde{\Sigma}_t = \Sigma \cup \{\infty_1, \ldots, \infty_L\}$ be the compactification of Σ_t. We have $g(\infty_k) = 0$ if k is odd and $g(\infty_k) = \infty$ is k is even.

Canonical homological basis

Canonical homological basis of Σ_t :

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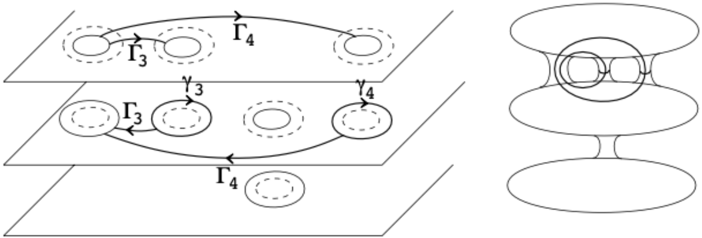

Figure: Canonical homology basis

Height differential

We need to define a meromorphic 1-form dh with (at most) simple poles at $\{\infty_1, ..., \infty_N\}$. Let $\hat{M}$ be a compact Riemann Surface of genus G.

- A canonical homology basis of $\hat{M}$ is a set of $2G$ closed curves γ_i, Γ_i for $1 \leq i \leq G$ such that γ_i intersects Γ_i with intersection number 1 and all other intersection numbers are zero.
- The space of meromorphic 1-forms with simple poles at given m points has complex dimension $G + m - 1$. One may prescribe the integrals along the curves $\gamma_1, ..., \gamma_G$ and the residues at the poles, with the only condition that the sum of the residues be zero.

Height differential

We can define the height differential dh_t as the unique meromorphic 1-form on Σ_t with poles at $\infty_1, ..., \infty_N$ such that

$$\forall i \in J, \int_{\gamma_{l,k}} dh_t = 2\pi i r_{l,k}, \quad 1 \leq l \leq L, \quad 1 < k \leq n_l, \quad \text{and } Res_{\infty_l} dh_t = -R_l$$

where $r_{l,k}$, are positive real numbers and $R_1, ..., R_N$ are real numbers whose sum is zero.

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$$\Omega_l := \{z \in \mathbb{C}_l : |v_{l,i}| \geq \varepsilon \quad \forall 1 \leq i \leq n_l \ \& \ |w_{l-1,j}| \geq \varepsilon \quad \forall 1 \leq j \leq n_{l-1}\},$$

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Moreover, as $t \to 0$, dh_t converges uniformly on a compact set of Ω_l to the form

$$\sum_{k=1}^{n_l} \frac{-r_{l,k} dz}{z - a_{l,k}} + \sum_{k=1}^{n_{l-1}} \frac{r_{l-1,k} dz}{z - b_{l-1,k}}. \tag{2}$$

Initial conditions

We want to find parameters $X = (t, a, b, \alpha, \beta, r, R)$ that solve the divisor conditions, period conditions, and regularity conditions. Let $(p^0_{l,k}, c^0_l, Q^0_l)$ be the neck positions, sizes and logarithmic growths of the given configuration. We will solve the period problem using the Implicit Function Theorem at the initial values of all parameter at point $t = 0$, denoted by X_0 as:

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$$t^\circ = 0, \quad R^\circ_l = Q^0_l,$$
$$\forall 1 \leq k \leq n_l,\ r^\circ_{l,k} = -\alpha^\circ_{l,k} = \beta^\circ_{l,k} = c^0_l$$
$$\forall 1 \leq k \leq n_l,\ a^\circ_{l,k} = \overline{b^\circ_{l,k}} = \begin{cases} \overline{p^0_{l,k}}, & l \text{ odd} \\ p^0_{l,k}, & l \text{ even.} \end{cases}$$

Solution of period problem

Divisor Condition: The zeros of dh in Σ_t must be precisely the zeros and poles of g, with the same multiplicity.

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On the other hand, whenever g has a zero or pole of multiplicity $m \geq 2$ at ∞_l, dh must have a zero or multiplicity $m - 2$; this corresponds to the planar ends.

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On the other hand, whenever g has a zero or pole of multiplicity $m \geq 2$ at ∞_l, dh must have a zero or multiplicity $m - 2$; this corresponds to the planar ends.

Proposition (Divisor condition [Traizet, 2002])

For (t, a, b, r, R) in a neighborhood of their initial values, there exist unique values for α and β, depending analytically on (t, a, b, r, R), such that the divisor conditions are satisfied. Moreover, at $t = 0$, we have $-\alpha_{l,k} = \beta_{l,k} = r_{l,k}$.

Continued...

Proposition (Vertical periods)

Assume that (α, β) are given by the previous proposition. For (τ, a, b, R) in a neighborhood of their initial values, there exists unique values for r, depending smoothly on (τ, a, b, R), such that the vertical period condition are satisfied over the curves $\Gamma_{l,k}$, $1 \leq l < L$ and $1 < k \leq n_l$. Moreover, at $\tau = 0$, we have $r_{l,k} = c_l$ for all $1 \leq k \leq n_l$ and $1 \leq l < L$, where c_l are defined from R_l by $c_0 = c_L = 0$ and $c_{l-1} n_{l-1} - c_l n_l = R_l$.

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Proposition (Horizontal periods)

Given a balanced and rigid configuration (p, Q) such that the map $Q \to W$ has rank 1. Assume that (α, β, r) are given by previous propositions. For τ in a neighborhood of 0, there exists unique values for a, b, and R, depending smoothly on τ, such that $\sum_l R_l = 0$ and the horizontal period condition are satisfied over the curves $\Gamma_{l,k}$ and $\gamma_{l,k}$, $1 \le l < L$ and $1 < k \le n_l$. Moreover, at $\tau = 0$, up to a translation in $_l$, we have $a_{l,k} = \overline{b_{l,k}} = \overline{p_{l,k}}$ if l is odd, $= p_{l,k}$ if l is even, and $R_l = Q_l$.

Singularities

The singularity set of a maxface is given by $\{|g(z)| = 1\}$. From the definition of Gauss map, the singularity set is given by the union of

$$\mathcal{S}_{l,k} = \{z \in \mathbb{C}_l : |v_{l,k}(z)| = t\} = \{z \in \mathbb{C}_{l+1} : |w_{l,k}(z)| = t\}$$

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for $1 \leq l \leq L-1, 1 \leq k \leq n_l$. Here, the domains of $v_{l,k}$ and $w_{l,k}$ are $\frac{t^2}{\epsilon} < |v_{l,k}| < \epsilon$ and $\frac{t^2}{\epsilon} < |w_{l,k}| < \epsilon$ respectively, satisfying $v_{l,k} \cdot w_{l,k} = t^2$.
For the family M_t with Weierstrass-Enneper representation, the characteristics a singular point is determined by three specific functions: $\alpha = \frac{g'}{gf}$, $\beta = \frac{g}{g'}\alpha'$, and $\eta = \frac{g}{g'}\beta'$.

Table

Re(α)	Im(α)	Re(β)	Im(η)	Description
$\neq 0$	$\neq 0$			$\Leftrightarrow$ cuspidal-edge
	$= 0$	$\neq 0$		$\Leftrightarrow$ swallowtail
	$= 0$	$= 0$	$\neq 0$	$\Leftrightarrow$ cuspidal butterfly
	$= 0$	$= 0$	$= 0$	$\Leftrightarrow$ p is a special singular point of type 1
$= 0$	$\neq 0$		$\mathrm{Im}(\beta) = 0$, $\mathrm{Re}(\eta) \neq 0$	$\Leftrightarrow$ p is a cuspidal S_1^-
	$\neq 0$		$\mathrm{Im}(\beta) = 0$, $\mathrm{Re}(\eta) = 0$	$\Leftrightarrow$ $p$is a special singular point of type 2
	$\neq 0$	$\mathrm{Im}(\beta) \neq 0$		$\Leftrightarrow$ p is a cuspidal crosscap
$\forall\ p, \neq 0$	$\equiv 0$	Compact Singular Set		$\Leftrightarrow$ Singular set is generalized cone-like singularity

Singularities

Theorem

On a maxfaces constructed above, for sufficiently small non-zero t

- *The singular set has N singular components, each being a curve around the waist of a neck.*
- *If the singular curve is not mapped to a single point (cone-like singularity), then all but finitely many singular points are cuspidal edges.*
- *If $R^1_{l,k} \neq 0$, then there are exactly four non-cuspidal singularities around the neck (l,k) and they are all swallowtails. Moreover, they tend to be evenly distributed on the waist as $t \to 0$.*

Infinite genus case

Configuration

We aim to construct a 1-parameter family of maxfaces M_t that look like infinite horizontal (spacelike) planes connected by small necks.

- For convenience, let us label each horizontal plane by an integer k,
- Let $n_k > 0$ for $k \in \mathbb{Z}$, denotes the number of necks connecting kth and $(k+1)$th spacelike planes,
- to each neck (k, l), we associate a complex number $p_{l,k} \in \mathbb{C}$ indicating its horizontal limit position at $t = 0$.

We write $p = (p_{k,i})_{1 \le i < n_k, k \in \mathbb{Z}}$. We call a sequence of complex numbers $(p_{k,i})_{1 \le i \le n_k, k \in \mathbb{Z}}$ as a configuration of type $(n_k)_{k \in \mathbb{Z}}$. let c_k be the real numbers which corresponds to the "size" of the necks at level k.

For each neck (k, i) in the configuration, we define the force $F_{k,i}$ exerted on $p_{k,i}$ as

$$F_{k,i} := \sum_{1 \le j \ne i \le n_k} \frac{2c_k^2}{p_{k,i} - p_{k,j}} - \sum_{j=1}^{n_{k+1}} \frac{c_k c_{k+1}}{p_{k,i} - p_{k+1,j}} - \sum_{j=1}^{n_{k-1}} \frac{c_k c_{k-1}}{p_{k,i} - p_{k-1,j}}.$$

Balanced and rigid configuration

Definition

A configuration is *balanced* if $F_{k,i} = 0$ for all $k \in \mathbb{Z}$ and $1 \leq i \leq n_k$, and *non-degenerate* if the differential of $F = (F_{k,i})_{k \in \mathbb{Z}, 1 \leq i \leq n_k}$ with respect to p has full rank.

Non-degeneracy condition motivates for the following translations in $(p_{k,i})$ as,

$$\begin{cases} u_{k,i} := (-1)^k (p_{k,i} - p_{k,1}), & 1 \leq i \leq n_k, k \in \mathbb{Z} \\ l_k := (-1)^k (p_{k,1} - p_{k-1,1}), & k \in \mathbb{Z}. \end{cases} \tag{3}$$

Thus, we get a new sequence $\mathbf{U}$:

$$\dots, l_k, u_{k,2}, \dots, u_{k,n_k}, l_{k+1}, u_{k+1,2}, \dots, u_{k+1,n_{k+1}}, l_{k+2}, \dots. \tag{4}$$

By definition, we always have $u_{k,1} = 0$ for all $k \in \mathbb{Z}$.

Infinite genus case [Dhochak, 2026]

Theorem

Consider a balanced, non-degenerate configuration $\mathbf{U}$ *of type* $(n_k)_{k\in\mathbb{Z}}$. *Further assume that*

1. *the sequence* $(n_k)_{k\in\mathbb{Z}}$ *is bounded,*
2. *the sequence* $\mathbf{U}$ *takes a finite number of values,*
3. *for all* $k \in \mathbb{Z}$, $\frac{1}{n_k}\sum_{i=1}^{n_k} p_{k,i} \neq \frac{1}{n_{k-1}} p_{k-1,i}$.

Then there exists a 1-parameter family $(M_t)_{0<t<\epsilon}$ *of complete, embedded (in a wider sense) maxfaces with infinitely many planar ends. Furthermore, each surface* M_t *is periodic (quasi-periodic) if and only if the configuration is periodic (quasi-periodic).*

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We say that the configuration is periodic if there exists a positive integer T such that for all $k \in \mathbb{Z}$, $n_{k+T} = n_k$, $l_{k+T} = l_k$ and $u_{k+T,i} = u_{k,i}$ for $2 \leq i \leq n_k$.

Height–2 family (type $(1, n, 1)$)

A configuration is given as

$$p_{0,1} = 0, \quad p_{1,j} = \iota + \cot\left(\frac{j\pi}{n+1}\right), \quad p_{2,1} = 2i \quad (1 \leq j \leq n).$$

- This finite configuration is balanced and non-degenerate [Morabito and Traizet, 2012].

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- For $n = 2$, the configuration corresponds to a Lorentzian Wei-type surface.
- By concatenation of a sequence of finite balanced and non-degenerate configuration method [Morabito and Traizet, 2012], we obtain an infinite singly–periodic configuration with alternating layers of n necks and 1 neck of type $(n_k)_{k\in\mathbb{Z}} = (\ldots, 1, n, 1, n, 1, \ldots)$ with period 2 where $p_{k+2,i} = p_{k,i} + 2\iota$.

Singularities

Singularities: From the equation [5], direct calculation shows that in this configuration

- $R_{k,1}^{(1)}(\theta) = 0$, for even k,
- $R_{k,i'}^{(1)}(\theta) = \frac{6\cos 2\theta}{n\left(1+\cot^2(\frac{i'\pi}{n+1})\right)^2}\left(\cot\frac{i'\pi}{n+1} - c\right) \neq 0$, when k is odd.

Therefore, even-layer necks have $R^{(1)} \equiv 0$ and their singular curve might degenerate to a cone point.
On the contrary, in odd layers $R_{k,i'}^{(1)} = 0$ only at isolated points. By [Chen et al., 2024, Theorem 2.3], such nonzeros of $R_{k,i'}^{(1)}$ implies exactly four swallowtail singularities around that neck. Therefore, every neck in an odd layer produces four swallowtails.

Continued...

Following the notation of [Chen et al., 2024], set

$$R_{k,i}^{(r)}(\theta) := \begin{cases} \mathrm{Im}\Big(e^{(r+1)i\theta}\overline{\mathrm{Res}_{p_{k,i}-p_{k-1,1}}\frac{\omega_k^{r+2}}{dz}} \;-\; e^{-(r+1)i\theta}\mathrm{Res}_{p_{k,i}-p_{k,1}}\frac{\omega_{k+1}^{r+2}}{dz}\Big), & k \text{ odd},\\ \mathrm{Im}\Big(e^{(r+1)i\theta}\mathrm{Res}_{p_{k,i}-p_{k-1,1}}\frac{\omega_k^{r+2}}{dz} \;-\; e^{-(r+1)i\theta}\overline{\mathrm{Res}_{p_{k,i}-p_{k,1}}\frac{\omega_{k+1}^{r+2}}{dz}}\Big), & k \text{ even}, \end{cases} \tag{5}$$

and

$$\omega_k = \sum_{i=1}^{n_{k-1}} \frac{c_{k-1}}{z-(p_{k-1,i}-p_{k-1,1})} - \sum_{i=1}^{n_k} \frac{c_k}{z-(p_{k,i}-p_{k-1,1})}.$$

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Thank You!!