Diffeology: definitions and examples

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(Why) beyond manifolds

"Manifolds are fantastic spaces. It's a pity that there aren't more of them."[1]

Definition (Informal)

A (smooth) manifold is a space that is modelled by a Euclidean space $\mathbb{R}^n$.

Such as spheres, tori, Möbius bands, Klein bottles.

- We have many tools to study them: Morse theory, Cartan calculus, de Rham / singular cohomology, etc.
- And many structures to put on them: Riemannian metrics, symplectic structures, foliations, etc.

[1] https://ncatlab.org/nlab/show/generalized+smooth+space

(Why) beyond manifolds

Yet many relevant spaces are not manifolds.

- The quotient of a manifold need not be a manifold.
- A subset of a manifold need not be a manifold.
- A space of maps between manifolds need not be a manifold.

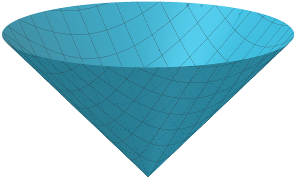
Diffeology, higher geometric structures, and infinite-dimensional manifolds are generalizations of manifolds that can handle these singular situations.

(Why) beyond manifolds

Quotients

The **cone orbifold** C_k is the quotient of the plane $\mathbb{R}^2 \cong \mathbb{C}$ by a rotation:

$$\mathbb{Z}_k \circlearrowright \mathbb{C}, \quad r \cdot z := e^{\frac{2\pi i r}{k}} z$$



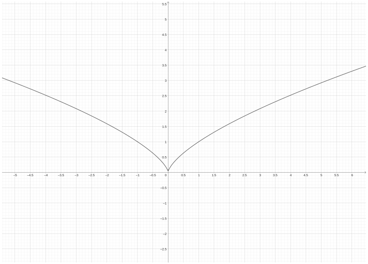
- Topologically: $C_k \underset{\text{top}}{\cong} \mathbb{R}^2$ for all k.
- Smoothly: C_k is not a smooth (quotient) manifold if $k > 1$.

(Why) beyond manifolds

Subsets

The **semi-cubic** S is the subset of the plane $\mathbb{R}^2$

$$S \coloneqq \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^3\}.$$



- Topologically: $S \underset{\text{top}}{\cong} \mathbb{R}$.
- Smoothly: S is not a (immersed sub-)manifold.

The set of **smooth maps** between manifolds is

$$C^\infty(M,N) := \{f\colon M \to N \mid f \text{ is infinitely differentiable}\}.$$

No picture!

- Topologically: $C^\infty(M,N)$ is a Fréchet[2] manifold if M is compact.
- Smoothly: $C^\infty(M,N)$ is not a Fréchet manifold if M is not compact.

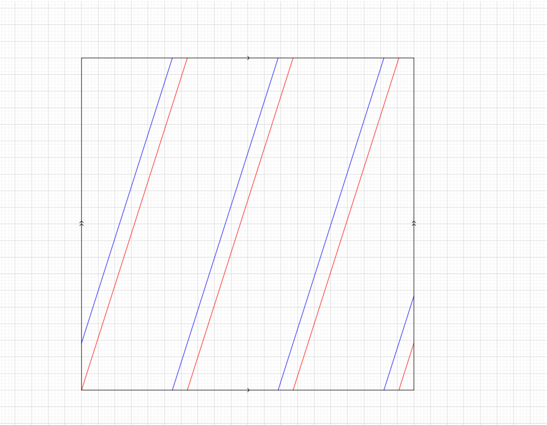
[2] infinite-dimensional

A diffeology is a structure on a set.

Space	Topology	Diffeology
Orbifold cone	$\mathbb{R}^2$	$C_k \underset{\text{dflg}}{\ncong} C_l$; sees the angle
Semi-cubic	$\mathbb{R}$	$S \underset{\text{dflg}}{\cong} \mathbb{R}$; does not see the cusp
$C^\infty(M, N)$	subtle	$TC^\infty(M, N) \underset{\text{dflg}}{\cong} C^\infty(M, TN)$

Another example: the **irrational torus** T_α, for an irrational number α, is

$$T_\alpha := \mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z}) \underset{\text{dflg}}{\cong} T^2/(\text{line of slope } \alpha).$$



The red and blue segments are a single connected curve in the torus.

Space	Topology	Diffeology
Orbifold cone	$\mathbb{R}^2$	$C_k \underset{\text{dflg}}{\cong} C_l$ iff $k = l$; sees the angle
Semi-cubic	$\mathbb{R}$	$S \underset{\text{dflg}}{\cong} \mathbb{R}$; does not see the cusp
$C^\infty(M, N)$	subtle	$TC^\infty(M, N) \underset{\text{dflg}}{\cong} C^\infty(M, TN)$

The irrational torus T_α has trivial topology, but

Theorem (Donato–Iglesias-Zemmour [DonIgl85])

$T_\alpha \underset{dflg}{\cong} T_\beta$ if and only if α and β are related by an integral homography.[a]

[a] i.e., $\alpha = \frac{a\beta+b}{c\beta+d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2; \mathbb{Z})$.

A **smooth n-dimensional manifold structure** on a set M is determined by an atlas consisting of **charts**[3]

$$\mathcal{A} = \{\varphi\colon \mathcal{O} \to M \mid \mathcal{O} \subseteq \mathbb{R}^n \text{ is open}\}.$$

Charts give:

- the local model, i.e., $\varphi(\mathcal{O}) \underset{\text{smooth}}{\cong} \mathcal{O}$;
- the smooth maps, i.e., $C^\infty(U, M)$ for each open $U \subseteq \mathbb{R}^k$

Diffeology removes the first feature, and keeps the second.

[3] charts typically go the other direction; these are "local parametrizations."

Definition (Souriau [Sour80][a])

[a] cf.K. T. Chen [Chen77]

A **diffeology** on a set X is an assignment $\mathcal{D}$ that sends each open $U \subseteq \mathbb{R}^k$, for each k, to a collection of maps $\mathcal{D}_U \subseteq \underline{\mathcal{S}\text{et}}(U, X)$, called **plots**, such that

- (concreteness) every constant map $U \to X$ is in $\mathcal{D}_U$, for each U;
- (pre-sheaf) if $f \colon V \to U$ is smooth, then $f^*\mathcal{D}_U \subseteq \mathcal{D}_V$;
- (sheaf) if $f \colon U \to X$, and each $u \in U$ has a neighbourhood U' with $f|_{U'} \in \mathcal{D}_{U'}$, then $f \in \mathcal{D}_U$.

We could identify $\mathcal{D}$ with $\coprod_k \coprod_{U \subseteq \mathbb{R}^k} \mathcal{D}_U$.

Diffeological spaces form a category $\mathcal{D}\mathrm{flg}$, where $f\colon X \to Y$ is **smooth** if

$$f_*\mathcal{D}^X \subseteq \mathcal{D}^Y.$$

$\mathcal{D}\mathrm{flg}$ is complete, co-complete, and locally Cartesian closed.[4]

- Subsets and products inherit natural diffeologies.
- Quotients and coproducts inherit natural diffeologies.
- The slice categories $\mathcal{D}\mathrm{flg}/X$ are Cartesian closed.

Every manifold is naturally a diffeological space: equip M with $\mathcal{D}^M$:

$$\mathcal{D}^M_U := \{p\colon U \to M \mid p \text{ is smooth in the usual sense}\}.$$

The functor $M \mapsto \mathcal{D}^M$ is full and faithful.

[4] It is a quasitopos.

A diffeological structure induces the **D-topology**:

$$D(X, \mathcal{D}) := \text{ the finest topology for which each plot is continuous.}$$

- D is left adjoint to the continuous diffeology functor

$$\mathcal{D}_U^{C(T)} := \underline{\mathcal{T}\mathrm{op}}(U, T).$$

  In other words, $\underline{\mathcal{T}\mathrm{op}}(D(X), T) = \underline{\mathcal{D}\mathrm{flg}}(X, C(T))$.
- $\mathcal{D}$flg is a concrete category over $\mathcal{T}$op; i.e., D is faithful.
- $D(M, \mathcal{D}^M) \underset{\text{top}}{\cong} M$.

Every set[5] X carries two distinct diffeologies.

- The **discrete** diffeology consists of only locally-constant plots,

$$\mathcal{D}_U^{\text{discrete}} := \{p\colon U \to X \mid p \text{ is locally-constant}\}.$$

  Its D-topology is discrete.

- The **coarse** diffeology consists of all maps.

$$\mathcal{D}_U^{\text{coarse}} := \underline{\mathcal{S}\text{et}}(U, X).$$

  Its D-topology is coarse.

Every other diffeology sits between these two.

[5] with more than two elements

Definition

The **quotient** diffeology on $X/\sim$ consists of maps $p\colon U \to X/\sim$ such that: for each $r \in U$,

$$\begin{tikzcd} & & X \arrow[d, "\pi"] \\ \exists(V \ni r) \arrow[r, hook] \arrow[urr, "\exists q"] & U \arrow[r, "p"] & X/\sim. \end{tikzcd}$$

$D(X/\sim) \underset{\text{top}}{\cong} D(X)/\sim$. This is because D is a left adjoint functor.

Consider irrational tori T_α and T_β.

- A diffeological diffeomorphism $f\colon T_\alpha \to T_\beta$ pulls back to a plot $f \circ \pi_\alpha\colon \mathbb{R} \to T_\beta$.
- This plot locally lifts about 0 to a map F fitting in the diagram

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d, "\pi_\alpha"] & \mathbb{R} \arrow[d, "\pi_\beta"] \\ T_\alpha \arrow[r, "f", "\cong"'] & T_\beta \end{tikzcd}$$

- One shows that F must be an equivariant embedding, which exists only if α and β are related by an integral homography.

An **orbifold** is a diffeological space that is locally diffeomorphic to:

$$\mathbb{R}^n/\Gamma, \text{ where } \Gamma \text{ is a finite group of linear transformations.}$$

Proposition (Iglesias-Zemmour–Karshon–Zadka [IglKarZad10])

The usual category of orbifolds embeds into $\mathcal{D}$flg.

Irrational tori **quasifolds**, where we replace "finite" with "countable."

Subset diffeology

Definition

Definition

The **subset** diffeology on $A \subseteq X$ consists of maps $p\colon U \to A$ such that $p\colon U \to X$ is smooth.

- $D(A)$ is not in general diffeomorphic to $A \subseteq D(X)$.
- For example, $\mathbb{Q} \subseteq \mathbb{R}$ does not have discrete sub-topology, but it does have discrete sub-diffeology, so $D(\mathbb{Q} \subseteq \mathbb{R})$ is discrete.
- Topological discreteness is weaker than diffeological discreteness.

Subset diffeology

Example

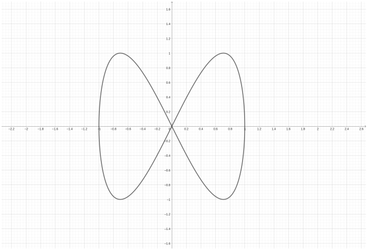
Theorem (Joris [Jor82], cf. Karshon–M.–Watts [KarMiyWat24])

The map

$$\mathbb{R} \to \mathbb{R}^2, \quad t \mapsto (t^3, t^2)$$

is a diffeological diffeomorphism onto its image (the cusp S). Equivalently $f : \mathbb{R} \to \mathbb{R}$ is smooth if and only if f^2 and f^3 are smooth.

Conversely, the "figure-eight" in the plane, with its subset diffeology, is not diffeomorphic to $\mathbb{R}$.



Definition

Let A be a subset of a manifold M.

- A is a **diffeological** submanifold if $(A, \mathcal{D}^A)$ is a manifold.[a]
- A is furthermore **weakly-embedded** if $\iota_A \colon (A, \mathcal{D}^A) \to M$ is an immersion.

[a] i.e. it is locally diffeomorphic to some $\mathbb{R}^n$.

- Stefan showed the leaves of a foliation are weakly-embedded [**Stef74**].
- Example: the orbits of the $\mathbb{R}$ action on T^2 by

$$\lambda \cdot [x, y] := [x + \lambda, y + \alpha\lambda] \text{ for fixed irrational } \alpha.$$

- Non-example: the figure-eight.

A **manifold with boundary** is a diffeological space that is locally diffeomorphic to a half-space

$$\{x_1, \ldots, x_k \geq 0\} \subseteq \mathbb{R}^n.$$

Proposition (Gürer–Iglesias-Zemmour [GuerIgl19])

The usual category of manifolds with boundary embeds into $\mathcal{D}flg$

Diffeology similarly gives an ambient category for manifolds with corners.

Functional diffeology

Definition

Definition

The **functional** diffeology on $C^\infty(X, Y)$ is

$$\mathcal{D}_U := \{p\colon U \to C^\infty(X, Y) \mid \hat{p}\colon U \times X \to Y,\ (r, x) \mapsto p_r(x) \text{ is smooth}\}$$

This was Chen's original motivation for his structures. Note: we use the product diffeology:

$$(\mathcal{D}^{\prod_\alpha X_\alpha})_U := \left\{(p_\alpha)\colon U \to \prod_\alpha X_\alpha \mid p_\alpha \in \mathcal{D}_U^{X_\alpha} \text{ for all } \alpha\right\}$$

Example

For M a manifold, $\mathrm{Diff}(M)$, with its sub-diffeology, is a diffeological group.

This was Souriau's original motivation for diffeology.

If $\pi\colon F \to X$ is a map, we can equip

$$\Gamma(F) := \{\sigma\colon X \to F \mid \pi \circ \sigma = 1_X\}$$

with the sub-diffeology.

- If $V \to M$ is a vector bundle, then $D(\Gamma(V))$ is the usual Fréchet (vector space) topology ([**Wal12**]).
- If $F \to M$ is a fiber bundle over compact M, then $D(\Gamma(F))$ is the usual Fréchet (manifold) topology.
- In complete generality, $\Gamma(F)$ does not admit a Fréchet manifold structure.

Definition

A **differential k-form** on X is an assignment α of each plot $p \in \mathcal{D}_U$ to a differential form $p^*\alpha \in \Omega^k(U)$, such that for any smooth map $F\colon V \to U$, we have

$$F^*(p^*\alpha) = (p \circ F)^*\alpha.$$

- The assignment $X \mapsto \Omega(X)$ is a functor into the category of differential graded algebras.
- Equivalently, Ω is the right Kan extension of the de Rham functor

  $$\Omega\colon \mathcal{E}\mathrm{ucl} \to \mathcal{D}\mathrm{GA},$$

  along the embedding $y\colon \mathcal{E}\mathrm{ucl} \to \mathcal{D}\mathrm{flg}$.

Definition

The **smooth homotopy group** of X at a point x is the group $\pi_1(X,x)$ of smooth stationary paths $C^\infty_{\mathsf{st}}([0,1],X)$, modulo smooth homotopy, with group operation concatenation.

- γ stationary means: there exists $\varepsilon > 0$ such that $\gamma|_{[0,\varepsilon)}$ and $\gamma|_{(1-\varepsilon,1]}$ are constant.
- Stationary allows the concatenation $\gamma * \gamma'$ to remain a smooth map.
- $\pi_1(M,x)$ is the usual fundamental group.

There are several[6] proposals for tangent functors on $\mathcal{D}$flg.

Maps in	Maps out	Mixed
Losik '92	Iglesias-Zemmour '13	Souriau '80
Hector '95	Christensen-Wu '16	Torre '98
Christensen-Wu '16	Taho '24	Leslie '03
Taho '25		Dugmore-Ntumba '07
		Vincent '08
		Stacey '13

See also Blohmann '24, Goldammer-Welker '21, Magnot '12

[6] infinitely many! [**Tah25**]

- Denote by $\hat{T}$ the tangent functor on $\mathcal{E}$ucl, with $\hat{T}U := U \times \mathbb{R}^n$.
- $y\colon \mathcal{E}\mathrm{ucl} \to \mathcal{D}\mathrm{flg}$ is the functor that realizes U as a diffeological space.

Definition

We define $T\colon \mathcal{D}\mathrm{flg} \to \mathcal{D}\mathrm{flg}$ as the left Kan extension of $y\hat{T}$ along y.

$$\begin{tikzcd} & \mathcal{D}\mathrm{flg} \arrow[d, "T"] \\ \mathcal{E}\mathrm{ucl} \arrow[ur, "y"] \arrow[r, "y\hat{T}"'] & \mathcal{D}\mathrm{flg} \end{tikzcd}$$

Proposition (Blohmann [**Bloh24**])

For a fiber bundle $F \to M$, we have $T\Gamma(F) \underset{dflg}{\cong} \Gamma(VF)$. In a simpler situation, $TC^\infty(M, N) \underset{dflg}{\cong} C^\infty(M, TN)$.

An unfamiliar example

The wire plane

We can equip $\mathbb{R}^2$ with the **wire** diffeology,

$$\mathcal{D}^{\text{wire}}_U := \{p\colon U \to \mathbb{R}^2 \mid p \text{ locally factors through smooth curves}\}.$$

Call $(\mathbb{R}^2, \mathcal{D}^{\text{wire}}) := W$. Explicitly, plots of W satisfy: for each $r \in U$,

$$\begin{tikzcd} & \mathbb{R} \arrow[dr, "\exists c"] & \\ \exists(V \ni r) \arrow[ur, "\exists F"] \arrow[rr, "p|_V"] & & \mathbb{R}^2 \end{tikzcd}$$

- $D(W) \underset{\text{top}}{\cong} \mathbb{R}^2$ [Iglesias-Zemmour [**Igl13**]].
- $C^\infty(W, \mathbb{R}) \underset{\text{set}}{\cong} C^\infty(\mathbb{R}^2, \mathbb{R})$ (by Boman's theorem).
- $\pi_1(W, (0,0))$ is uncountably generated.

Thank You!

References I