Higher and infinite-dimensional structures in geometry

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Last time: $\mathcal{D}flg$ category of diffeological spaces
- complete $\to$ includes all subsets
- cocomplete $\to$ includes all quotients   cf. Lie groupoids
- Cartesian closed $\to$ includes all mapping spaces   cf convenient calculus

Reminder: $X \in \mathcal{D}flg$ is a set equipped with a diffeology $\mathcal{D}$

$$\text{plots} \in \mathcal{D}_U \subseteq \underline{Set}(U, X), \quad \text{for each } U \subseteq \mathbb{R}^n, \text{ for each } n \geq 0$$

Equivalently, $\mathcal{D}$ is a concrete sheaf on $\mathcal{E}ucl$ $\begin{cases} \text{objects } U \subseteq \mathbb{R}^n, \ n \geq 0 \\ \text{morphisms } V \xrightarrow{f} U \text{ smooth} \end{cases}$

Let M be a (smooth finite-dimensional) manifold, $\sim$ equivalence relation

$$\mathcal{D}^{M/\sim}_U := \left\{ p : \underset{U \subseteq \mathbb{R}^n}{U} \to M/\sim \ \middle| \ \forall r \in U, \ \exists \begin{cases} V \text{ nbhd in } U \text{ of } r \\ V \xrightarrow{q} M \end{cases} \text{ s.t. } \begin{tikzcd} & M \arrow[d, "\pi"] \\ V \arrow[ur, "q"] \arrow[r, "p|_V"'] & M/\sim \end{tikzcd} \text{ commutes} \right\}$$

<u>Results</u>

$$\mathbb{R}^2/\mathbb{Z}_m \underset{\mathcal{D}flg}{\cong} \mathbb{R}^2/\mathbb{Z}_n \quad \text{iff } m = n$$

$$\mathbb{R}^n/O(n) \underset{\mathcal{D}flg}{\cong} \mathbb{R}^m/O(m) \quad \text{iff } m = n$$

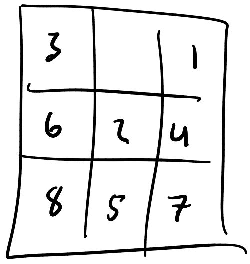
$$\mathbb{R}/(\mathbb{Z}+\alpha\mathbb{Z}) \underset{\mathcal{D}flg}{\cong} \mathbb{R}/(\mathbb{Z}+\beta\mathbb{Z}) \quad \text{iff } \alpha \equiv \beta \text{ mod } GL(2,\mathbb{Z})$$

$\{U_i\}$ cover of M

$$\bigsqcup U_i \Big/ (x,j) \sim (x,i) \underset{\mathcal{D}flg}{\cong} M$$

A Lie groupoid approach to quotients

Lie groupoids axiomatize "local symmetry"



Rubik's cube $\updownarrow$ group

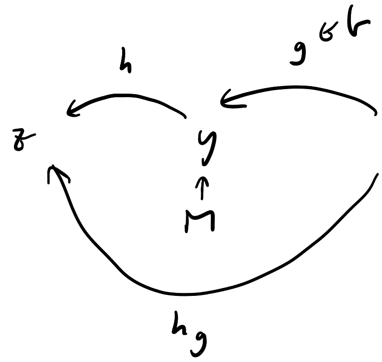
15 puzzle $\updownarrow$ groupoid

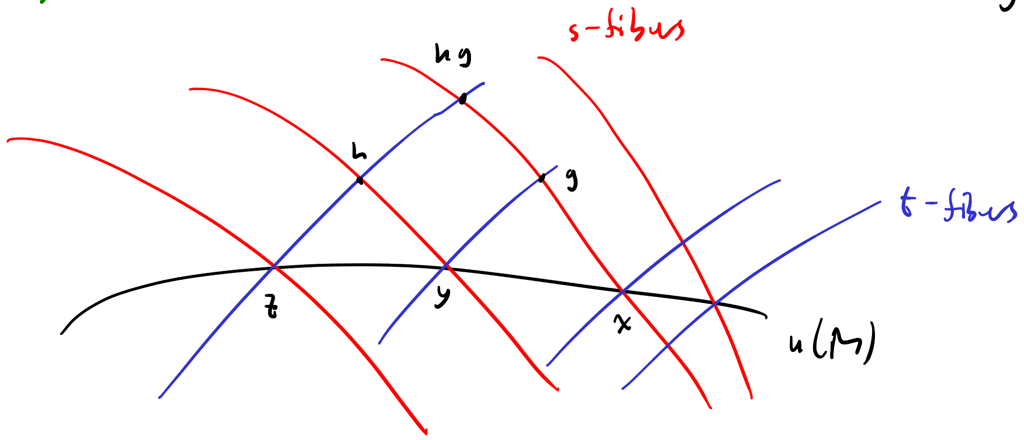
<u>Definition</u> A Lie groupoid is

- A category $G \rightrightarrows M$ such that every morphism is invertible
  Groupoid
- G, M equipped with manifold structures, where all structure maps are smooth
  Smooth

Furthermore, require source and target are submersions

In detail, have
source $s: G \to M$
target $t: G \to M$
multiplication $m: G\ {}_s\!\times_t G \to G$
$(g,h) \mapsto gh$
unit $u: M \to G$
groupoid $\Rightarrow$ inversion $\iota: G \to G$





Examples

- H Lie group $\rightsquigarrow$ $H \rightrightarrows \{*\}$ $\quad * \xleftarrow{h} *$, $\quad m(g,h) = gh$
  Orbit space is $\{*\}$
- M manifold $\rightsquigarrow$ $M \rightrightarrows M$ $\quad x \xleftarrow{x} x$ $\quad x \cdot x = x$
  Orbit space is M
- $M \circlearrowleft H$ $\rightsquigarrow$ $M \times H \rightrightarrows M$ $\quad x\cdot h \xleftarrow{(x,h)} x$ $\quad (y,g)(x,h) = (x, x\cdot hg)$
  $y = x\cdot h$

  Orbit space is M/H

·) $\{U_i\}$ cover of M $\rightsquigarrow$ $\coprod U_{ji} \rightrightarrows \coprod U_i$ $\quad (x,j) \xleftarrow{(x,ji)} (x,i)$

Orbit space is M

Every Lie groupoid has an orbit space

$$M/G := M\big/(x\sim y \text{ if } \exists g \text{ s.t. } y \xleftarrow{g} x)$$

Lie groupoids as atlases for quotients

Observe
$\begin{matrix} M \\ \downdownarrows \\ M \end{matrix}$ and $\begin{matrix} \coprod U_{ji} \\ \downdownarrows \\ \coprod U_j \end{matrix}$ are not isomorphic as Lie groupoids

But there is a "natural" functor
$$\begin{tikzcd} \coprod U_{ij} \arrow[r, hook, "\iota"] \arrow[d, shift left] \arrow[d, shift right] & M \arrow[d, shift left] \arrow[d, shift right] \\ \coprod U_i \arrow[r, hook, "\iota_0"] & M \end{tikzcd}$$

We require a more flexible notion of isomorphism

Definition A Morita map is a functor
$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, shift left] \arrow[d, shift right] & H \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, "\varphi_0"] & N \end{tikzcd}$$
that is

·) essentially surjective $M \,{}_{\varphi_0}\!\times_t H \xrightarrow{s\circ pr_2} N$ is a surjective submersion

- $\forall y \in N$, $\exists x \in M$ and $h\in H$ s.t. $\varphi_0(x) \xleftarrow{h} y$

·) fully faithful
$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "(s,t)"'] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[d, "(s,t)"] \\ M\times M \arrow[r, "\varphi_0\times\varphi_0"] & N\times N \end{tikzcd}$$
is a pullback

- $\forall (x',x) \in M\times M$, and $\varphi_0(x') \xrightarrow{h} \varphi_0(x)$, $\exists!\ x' \xleftarrow{g} x$ s.t. $\varphi(g)=h$

Definition G is Morita Equivalent (ME) to H if there exists

$$\begin{tikzcd} & K \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ G & & H \end{tikzcd}$$
Morita maps

Examples

- $\mathbb{Z}_n \curvearrowright \mathbb{R}^2$ ME $\mathbb{Z}_m \curvearrowright \mathbb{R}^2$ iff $m=n$
- $O(n) \curvearrowright \mathbb{R}^n$ ME $O(m) \curvearrowright \mathbb{R}^m$ iff $m=n$
- $\mathbb{Z}+\alpha\mathbb{Z} \curvearrowright \mathbb{R}$ ME $\mathbb{Z}+\beta\mathbb{Z} \curvearrowright \mathbb{R}$ iff $\alpha \equiv \beta \bmod GL(2,\mathbb{Z})$
- $\sqcup U_{ij}$ is ME to M

Theorem Let $\Gamma \curvearrowright \mathbb{R}^n$ effectively. Suppose one of the following hold

1) Γ finite
2) Γ countable $\leq \mathrm{Aff}(\mathbb{R}^n)$
3) Γ countable $\leq \mathrm{Iso}(\mathbb{R}^n, g)$ any metric

Suppose $\Gamma' \curvearrowright \mathbb{R}^m$ is similar.

Then $\Gamma \curvearrowright \mathbb{R}^n$ is ME to $\Gamma' \curvearrowright \mathbb{R}^m$ iff $\mathbb{R}^n/\Gamma \underset{\text{orbfly}}{\cong} \mathbb{R}^m/\Gamma'$

1) Iglesias-Zemmour, Laffineur '17
2) Iglesias-Zemmour, Prato '20 cf. Karshon, M '23
3) M- '24

non-example! - $O(n) \curvearrowright \mathbb{R}^n$ is not ME $SO(n) \curvearrowright \mathbb{R}^n$
despite having same orbit space

- $H \rightrightarrows \{*\}$, $G \rightrightarrows \{*\}$

Fact! H is ME to $G \iff H \underset{\text{Lie}}{\cong} G$