Diffeological Differential Geometry

Topology and geometry seminar, IIIT Delhi

David Miyamoto, Queen's University

Last time .) Introduced Dflg

.) Introduced Lie groupoids $G \rightrightarrows M$

have some examples of Lie groupoids whose Morita class is determined by M/G

Brief interlude on infinite dimensional geometry

Definition A Fréchet space is a topological vector space E $\left\{ \begin{array}{l} + : E \times E \to E \\ \cdot : \mathbb{R} \times E \to E \end{array} \right.$ cts

whose topology is generated by a countable collection of seminorms,

s.t. E is Hausdorff and complete (Cauchy sequences converge)

example .) $\mathbb{R}^n$, n finite

.) Banach spaces (complete normed vector spaces)

There are two diffeologies we may consider on E

continuous linear functionals

1) $\mathcal{D}^{lin} := \Pi L(E, \mathbb{R}) = \Pi E^*$

$$\mathcal{D}^{lin}_U = \{ U \xrightarrow{p} E \mid l \circ p : U \to \mathbb{R} \text{ smooth } \forall\, l \in L(E, \mathbb{R}) \}$$

2) $$\mathcal{D}^{conv}_U = \{ U \xrightarrow{p} E \mid c := p \circ \tilde{c} : \mathbb{R} \to E \ \underline{\infty\text{-differentiable}} \ \forall \text{ smooth curves } \mathbb{R} \to U \}$$

c is ∞-differentiable if .) $c^{(1)} : \mathbb{R} \to E$, $t \mapsto \lim_{h \to 0} \frac{c(t+h) - c(t)}{h}$ exists

.) $c^{(n)}$ exists $\forall n \in \mathbb{N}$

For Fréchet spaces, $\mathcal{D}^{lin} = \mathcal{D}^{conv}$ (Kriegl-Michor 97)

For $E = \mathbb{R}^n$, this is Boman's theorem ('67)

↳ $\mathcal{D}^{lin}$ is standard diffeology, a priori $\mathcal{D}^{lin}_U \subseteq \mathcal{D}^{conv}_U$

Remark: For a general Hausdorff LCTVS E, if E is bornological,

then Kriegl-Michor call E convenient if $\mathcal{D}^{lin} = \mathcal{D}^{conv}$

Examples .) For a vector bundle $V \xrightarrow{\pi} M$ in Mfld,

D-top $\left(\Gamma(V) = \{ \sigma : M \to V \mid \pi \circ \sigma = 1_M \}, \text{ functional dflg} \right) =$ usual Fréchet top.

usual = equip V with some Riemannian metric, seminorms measure derivatives of σ

·) For $\Gamma(V)$, $D^{curv} = D^{im}$ and these coincide with functional diffeology

·) $C^k([0,1], M) = P^k(M)$ is a Banach space

A Singular Serre-Swan Theorem (A. Garmendia, Me, L. Ryvkin '25)

Classical Serre-Swan: Fix M connected smooth manifold

$$\Gamma : \left\{ \begin{matrix} \text{vector bundles} \\ V \to M \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} C^\infty(M)\text{-modules } Q \text{ that are} \\ \cdot)\ \text{finitely generated} \\ \cdot)\ \text{projective} \end{matrix} \right\}$$

$\exists\, R$ s.t. $Q \oplus R$ is free

$$V \longmapsto \Gamma(V)$$

- fiberwise addition
- pointwise $C^\infty(M)$-multiplication

is an equivalence of categories

- in particular, each such Q is isomorphic to some $\Gamma(V)$

What about non-projective Q?

Example: $M = \mathbb{R}$, $Q = (\mathbb{R}, +)$, $C^\infty(\mathbb{R})$-module structure $\underset{C^\infty(\mathbb{R})}{f} \cdot \underset{\mathbb{R}}{q} := f(0)\, q$

- Q is a projective $\mathbb{R}$-module
- Q is not a projective $C^\infty(\mathbb{R})$-module

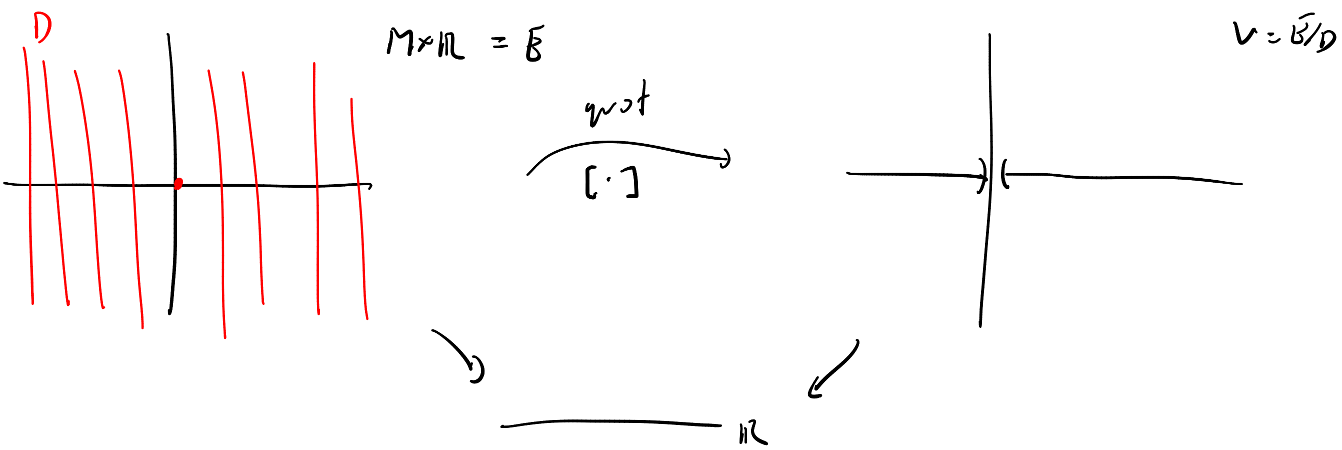
Seek $V \to \mathbb{R}$ s.t. $\Gamma(V) \cong \mathbb{R}$

- V should only have one non-trivial fibre that is 1-dimensional
- This is certainly not a vector bundle (violates local triviality)

We can find such a V in Dflg:

Take $V := (M \times \mathbb{R})/D$ where $D \subseteq M \times \mathbb{R}$, $D_t = \begin{cases} \mathbb{R} & \text{if } t \neq 0 \\ \{0\} & \text{if } t = 0 \end{cases}$

$$= (M \times \mathbb{R}) / \big((t, v) \sim (t, v') \text{ if } v - v' \in D_t \big)$$



Claim: $\Gamma(E/D) \cong \mathbb{R}$

Step 1) $\Gamma(E/D) \underset{\text{Dfly}}{\cong} \Gamma(E)/\Gamma(D)$

$$[s] \longleftarrow\!\shortmid\ s + \Gamma(D)$$

proof of surjectivity: given $\sigma : M \to E/D$, by definition of $\mathcal{D}^{E/D}$, there is some cover $\{U_i\}$ of M and lifts σ_i s.t.

$$\begin{tikzcd}
 & E \arrow[d] \arrow[dr, bend left, "\delta"] & \\
U_i \arrow[ur, "\sigma_i"] \arrow[r, "\sigma|_{U_i}"'] \arrow[rr, bend right, "\text{inclusion}"'] & E/D \arrow[r] & M
\end{tikzcd}$$

Can use partition of unity to glue σ_i into $\tilde{\sigma} \in \Gamma(E)$ s.t. $[\tilde{\sigma}] = \sigma$

Step 2) $\Gamma(E) \cong C^\infty(\mathbb{R})$

$\Gamma(D) \cong I_0 := \{ f \in C^\infty(\mathbb{R}) \mid f(0) = 0 \}$

and $C^\infty(\mathbb{R}) / I_0 \cong \mathbb{R}$

Note: Step 1) holds for E any vector bundle

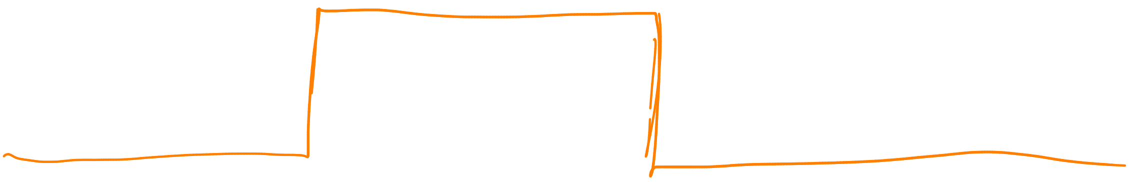
D any smooth singular subbundle

- a subset $D \subseteq E$ s.t. ·) D_x is a vector subspace of E_x
  ·) $\forall\, v \in D_x$, $\exists\, \sigma \in \Gamma(E)$ s.t. $\sigma(x) = v$, $\sigma(M) \subseteq D$

Definition: A bundle of diffeological vector spaces $V \to M$ is called

- have smooth $+ : V \times_M V \to V$
- smooth $\lambda : \mathbb{R} \times V \to V$

a VB-tepui of globally bounded rank if $V \cong E/D$ for some E, D as above



Theorem (GMR 25)

$$\Gamma : \left\{ \begin{array}{c} \text{VB-topoi of globally} \\ \text{bounded rank} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \overset{\text{Fréchet}}{C^\infty(M)}\text{-modules } Q \text{ that are} \\ \cdot)\ \text{finitely generated} \\ \cdot)\ \text{fibre determined} \end{array} \right\}$$

$$\begin{array}{ccc} V & \longmapsto & \Gamma(V) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ E/D & & \Gamma(E/D) \cong \Gamma(E)/\Gamma(D) \end{array}$$

is an equivalence of categories

note fibre-determined means $\forall\, q \in Q$,

$q = 0 \iff$ the image of q in $Q/I_x Q$ is 0 $\forall\, x \in M$

$I_x = \{ f \in C^\infty(M) \mid f(x) = 0 \}$

- a necessary condition for $Q \cong \Gamma(V)$

Lemma: A diffeological bundle of vector spaces $V \to M$

where · V and M are manifolds

· and $V \to M$ is a submersion

Then $V \to M$ is locally trivial