Basic notions of knot theory

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Def A (n oriented) knot (link) in $\mathbb{R}^3$ or in S^3 is a smooth embedding $K : S^1 \xrightarrow{\text{w/ orientation}} \mathbb{R}^3$ (or S^3)

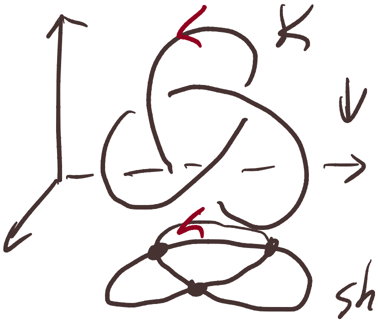
$L : S^1 \sqcup \cdots \sqcup S^1 \to$

as well as the image of embeddings.

Def Two (oriented) knots (or links) K_1 and K_2 are equivalent, $K_1 \sim K_2$

if $\exists\, H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ s.t. ① $H|_{\mathbb{R}^3 \times \{t\}} : \mathbb{R}^3 \times \{t\} \to \mathbb{R}^3$. orientation preserving homeo.

② $H|_{\mathbb{R}^3 \times \{0\}} = id_{\mathbb{R}^3}$, $H|_{\mathbb{R}^3 \times \{1\}}(K_1) = K_2$



shadow

overcrossing

a crossing

undercrossing

arc

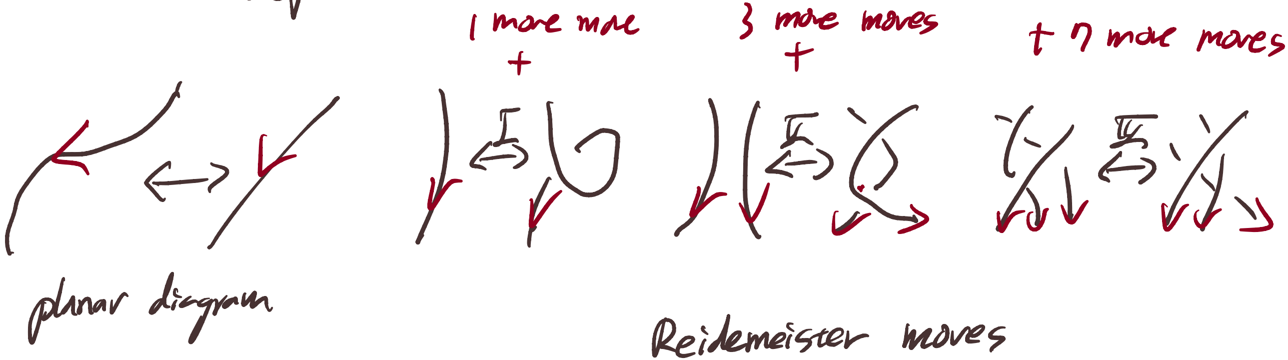
(link)

: a (knot) diagram D_K

Thm (1927, Reidermeister)

K_1, K_2: links, D_{K_1}, D_{K_2}: diagrams of K_1 and K_2, resp.

$K_1 \sim K_2$ iff D_{K_2} can be obtained from D_{K_1} by a finite seq. of the following local deformations



Remark Thanks to Reidemeister theorem, "topological" object is changed to "combinatoric" object.

Ex.

Trivial knot

trefoil

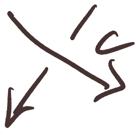
Figure-eight knot

Hopf link

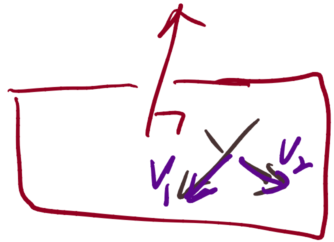
Borromean ring.

Def (crossing sign)

+1



−1 ← crossing sign of c



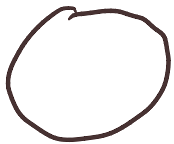
(v_1, v_2)

Main goal To distinguish knots and links in S^3.

Exa

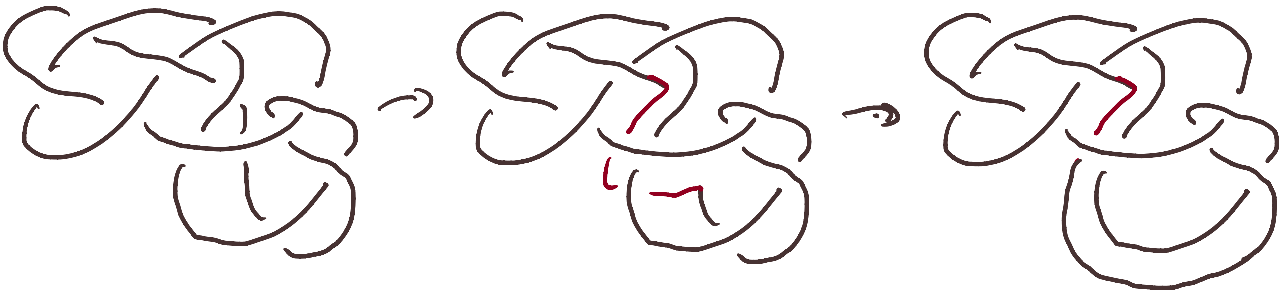
D_{K_1}

$\overset{?}{\simeq}$

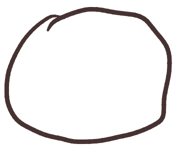


D_T

Answer. Yes.

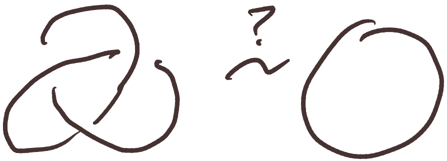


$\cdots \to$



i.e. $K_1 \simeq T$

Exa 2



Invariant of knots

$$
\begin{tikzcd}
\{\text{links}\}/\sim \arrow[r, "\Phi"] \arrow[d, leftrightarrow, "1-1"'] & \mathcal{C} \\
\{\text{knot diagrams}\}/\text{R.M.s} \arrow[ur, "\tilde{\Phi}"'] &
\end{tikzcd}
$$

Φ ← an invariant of knots & links

$\mathcal{C}$ ← a set of groups, numbers, polynomials, ...

$K_1 \sim K_2 \longmapsto \Phi(K_1) = \Phi(K_2)$

Coloring invariant

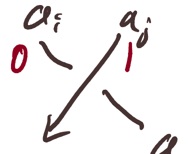
D: an oriented diagram, $A(D)$: the set of arcs of D.

$A(D) = \{a_1, a_2, a_3\}$

A proper $\overset{q\ (\text{prime number})}{3}$-coloring is a map

$$\rho : A(D) \longrightarrow \{0, 1, 2\} \Rightarrow \mathbb{Z}_q$$

satisfying

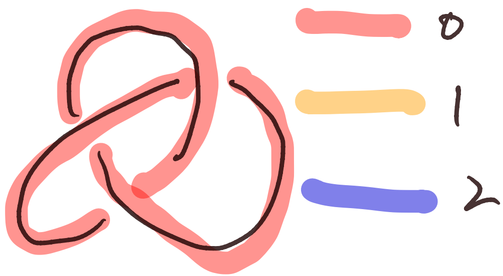


$\rho(a_i) = \rho(a_j) = \rho(a_k)$

or

$\rho(a_i) \neq \rho(a_j) \neq \rho(a_k)$

$\Rightarrow \rho(a_k) \equiv 2\rho(a_j) - \rho(a_i) \mod q$

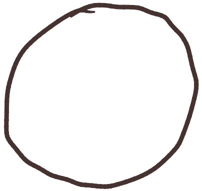


Trivial coloring

A (nontrivial) proper 3-coloring.

If D has a nontrivial proper $\overset{q}{3}$-coloring, then D is called $\overset{q}{3}$-colorable.

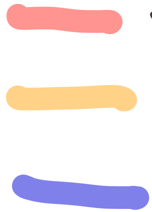
Exa



Not 3-colorable

Three trivial proper 3-coloring.

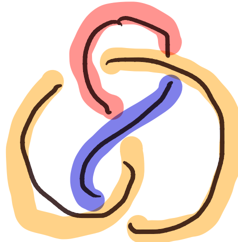
$\exists$ non-trivial proper 3-coloring. i.e 3-colorable.



0

1

2



$\nexists$ non-trivial proper 3-coloring, i.e. Not 3-colorable.

???

Lemma 3-colorability is preserved under R.M.

In other words, if $D \sim D'$ and $\exists p : A(D) \to \{0,1,2\}$, then

$\exists!\, p_{D'} : A(D') \to \{0,1,2\}$ corresponding to p_D

Idea of proof) Assume D' can be obtained from D by one of R.M.,

Case 1.

a

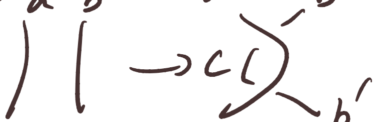
D $p(a) = l$

a

a'

D' $p'(a) = l = p'(a')$

Case 2 a b



$p(a) = l$

$p(b) = m$

a b c b'

$p'(a) = l$

$p'(b) = m = p'(b')$

If $l = m$, $p'(c) = l$.

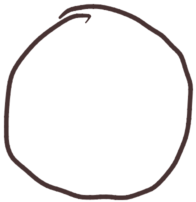
If $l \neq m$, $p'(c) = n$

Thm If $D \sim D'$, then

$\#\{\text{proper } \overset{7}{3}\text{-colorings of } D\} = \#\{\text{proper } \overset{9}{3}\text{-colorings of } D'\}$

⇕ 3-coloring invariant.

Exa



T

$\not\sim$

D

$\#\{\text{proper 3-colorings of } D\} = 3 \neq \#\{\text{proper 3-colorings of } D\} \geq 4.$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \quad 9 = 3^2$

Exa

$\mathbb{Z}_5$

$2\cdot 0 - 2 \equiv 3 \mod 5$

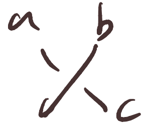
$a = 2\cdot 1 - 0 = 2 \mod 5$ i.e. it is 5-colorable.

Remark

$\#\{\text{proper } 5\text{-colorings of } D\} = 25 = 5^2$

Exer $\#\{\text{proper } q\text{-colorings of } D\} = q^k$.

Iden.



$c = 2b - a$

$a - 2b + c = 0$.

$$\Rightarrow \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{pmatrix} -2 & 1 & 1 & 0 \\ & & & \\ & & & \\ & & & \end{pmatrix} \end{array} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\in M_{4\times 4}(\mathbb{Z}_5)$

$a_2 - 2a_1 + a_3 = 0$

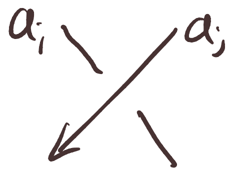
Since 5 is prime, $\mathbb{Z}_5$ is field.

So, we can apply linear algebra directely.

Exa

3-colorable.

Quandle and quandle coloring

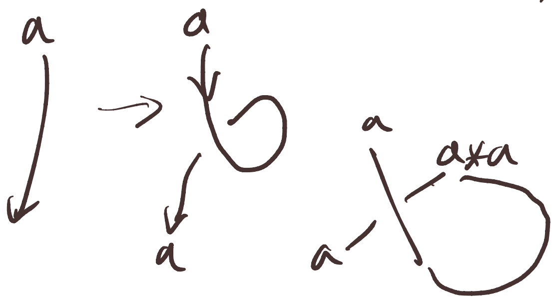


$a_k = 2a_j - a_i$
$= a_i * a_j$

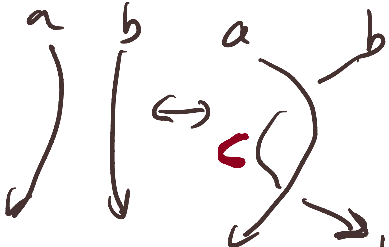
Q: a set w/ a binary operation $*$.

$\rho : A(D) \to Q$, $a_k = a_i * a_j$ $\forall c$: crossing.

To obtain a coloring invariant by $(Q, *)$, $(Q, *)$ should satisfy some conditions:

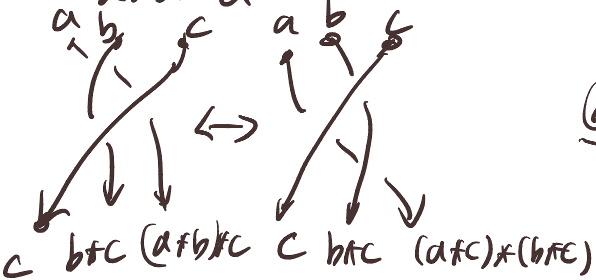


$a * a = a$



$b = c * a$.

$\forall a, b \in Q$, $\exists ! c$ s.t. $c * a = b$.

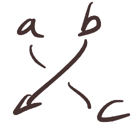


$(a*b)*c = (a*c)*(b*c)$

Def. A quandle is a set Q w/ bin. oper. $*$ s.t

1. $\forall a,\ a*a=a$
2. $\forall a,b,\ \exists!\, c$ s.t $c*a=b$
3. $\forall a,b,c\quad (a*b)*c=(a*c)*(b*c)$

A proper coloring of D by $(Q,*)$ is a map $A(D)\to Q$ s.t



$c=a*b$

Thm If $D\sim D'$, then $\exists$ 1-1 correspondence between

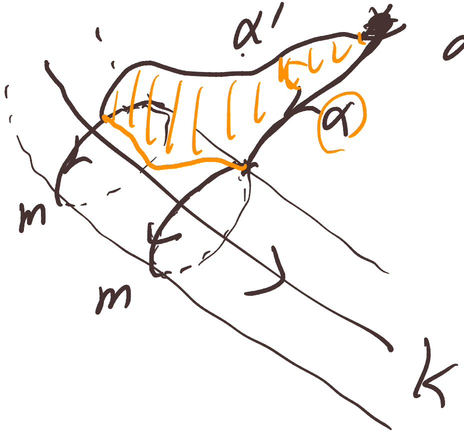
$$\{\text{proper coloring of } D \text{ by } Q\} \overset{1-1}{\longleftrightarrow} \{\text{proper coloring of } D \text{ by } Q\}$$

Exa.
- $(\mathbb{Z}_p,*)$ $a*b=2b-a$: a dihedral quandle.
- $(G,\cdot)$: a group, $\Rightarrow (G,*)$, $a*b=bab^{-1}$: a conjugation quandle.
- $(\mathbb{Z}[t^{\pm1}],*)$ $a*b=ta+(1-t)b$: Alexander quandle.

Knot quandle

tubular nbd of K.

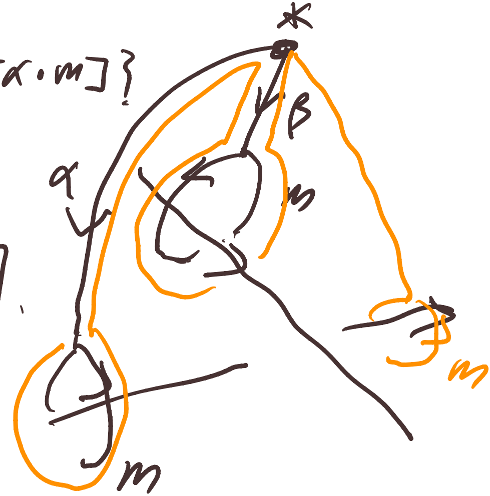
For a knot K, we consider $S^3 - N(K)$



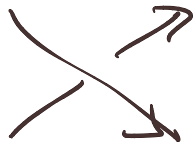
$\alpha \cdot m \sim \alpha' \cdot m$ $\quad Q(K) = \{[\alpha \cdot m]\}$

Define $[\alpha \cdot m] * [\beta \cdot m]$

$= [\beta \cdot m^{-1} \cdot \beta^{-1} \cdot \alpha \cdot m]$.



$\Rightarrow (Q(K), *)$: a quandle. $\Leftarrow$ knot quandle.



Lemma If $K \sim K'$, then $Q(K)$ and $Q(K')$ are isomorphic.

Remark

There is natural correspondence between $\pi_1(S^3 - N(K))$ and $Q(K)$

Thm Let K, K' be oriented knots.

If $Q(K) \cong Q(K')$, then $K \sim K'$ or $-K \sim K'$ or $K^* \sim K'$ or $-K^* \sim K'$.

where $-K$: ori. reversing

K^* : mirror image

So, knot quandle is called an almost complete invariant.