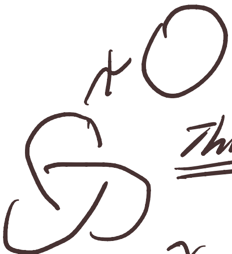
Kauffman bracket and the colored Jones polynomial.

Review

- A knot $K: S^1 \hookrightarrow \mathbb{R}^3$ (or S^3): a smooth embedding up to <u>ambient isotopy</u>.
  link $\quad S^1 \sqcup S^1 \sqcup \cdots \sqcup S^1$

  $H: \mathbb{R}^3 \times I \to \mathbb{R}^3$ s.t. $H(\cdot, t)$: homeo. on $\mathbb{R}^3$

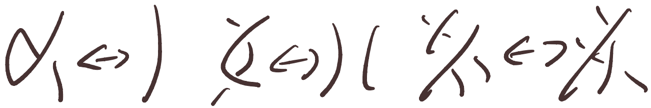
  - $H(\cdot, 0) = id.$, $H(\cdot, 1)(K_1) = K_2$.

- 

  diagram

  <u>Thm</u> Two links are equiv. iff. corresponding diag.s are equiv. modulo

  Reidemeister moves

- Invariants

$$\begin{tikzcd} \{\text{links}\}/\sim_{\text{ambient isotopy}} \arrow[r, "\Phi"] \arrow[d, leftrightarrow, "1-1"'] & \mathcal{C} \\ \{\text{diagrams}\}/\sim_{\text{RM.s}} \arrow[ur, "\tilde{\Phi}"'] & \end{tikzcd}$$

Ex. Quandle coloring invariant $\mathcal{C} = \mathbb{N} \cup \{\infty\}$

- Knot quandle $\quad \mathcal{C} = \{\text{groups}\}/_{\text{isom.}}$

Frame link

Def A framed knot K :  $\to \mathbb{R}^3$: a smooth embedding up to ambient isotopy.

link L : 

K $\quad K'$

$lk(K,K') = \frac{1}{2}(6\cdot(-1)+4\cdot -1)$
$= -1$

Def (Linking number) $L = K_1 \sqcup K_2$: oriented link.

$+1 \qquad -1$

crossing sign of c
$sgn(c)$

$lk(K_1, K_2) = \frac{1}{2}\sum sgn(c)$

c: crossing between different components

Linking number.

Exa

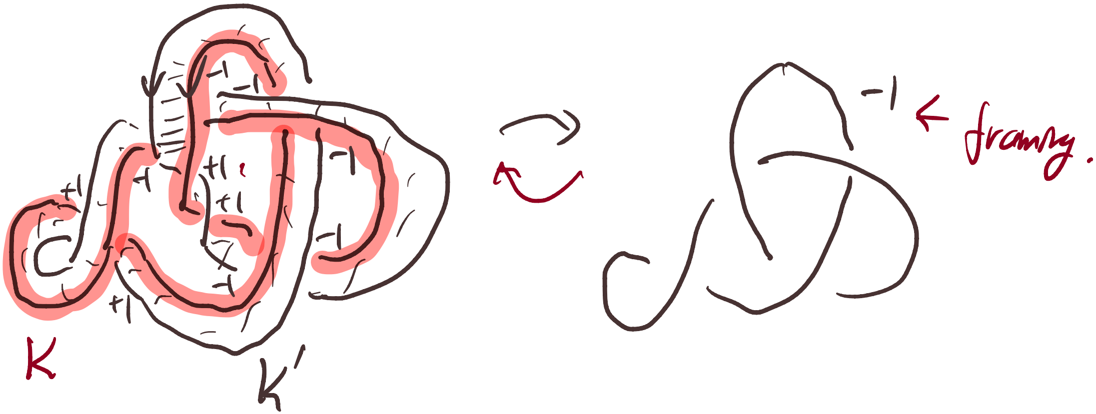
L

$K_1 \qquad K_2$

$lk(K_1,K_2) = \frac{1}{2}\sum sgn(c) = \frac{1}{2}(+1+1) = 1$

 $sgn = +1$

c: crossing between diff. comp.



$\{\text{framed links}\} \xleftrightarrow{1-1} \{\text{links w/ integer numbers}\}$

Remark

How to represent framed links by diagrams and which local moves are required?

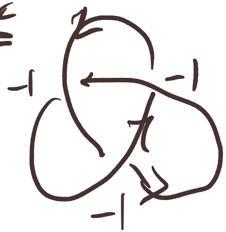
Blackboard framing

blackboard framing

$\longleftrightarrow$

$$lk(K, K') = \frac{1}{2} \cdot (6 \cdot (-1) + 2 \cdot 1)$$
$$= -2$$

Remark



$w(D) = -3$

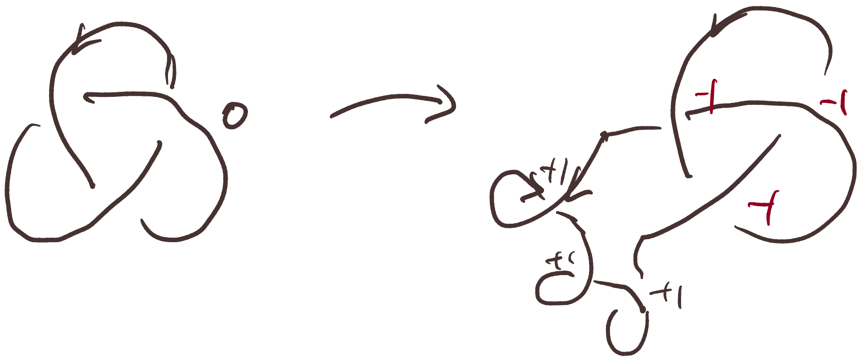
$lk(K, K') = -3$

$lk(K, K') = w(D)$: the writhe of D.

$= \sum \mathrm{sgn}(c)$

c: crossing of D.

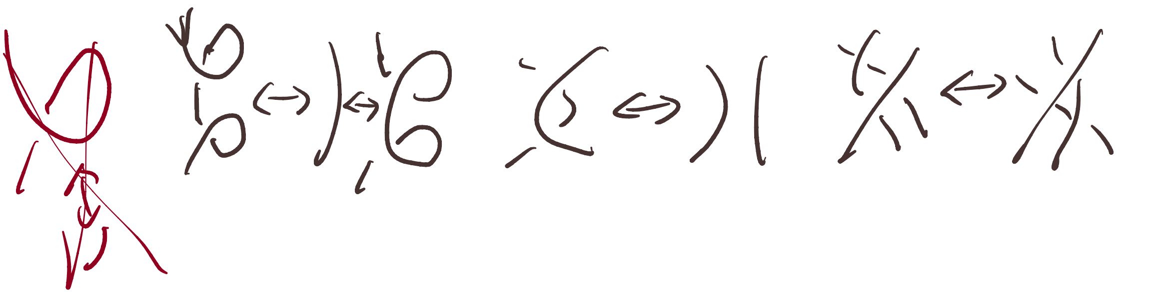
From the above observation,

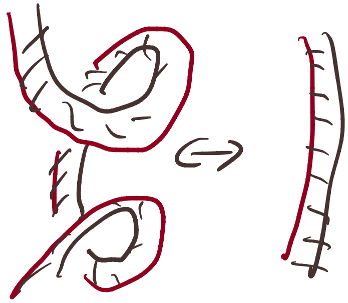


In this case, blackboard framing gives linking number -3.

A diagram corresponding to a framed knot.

Prop Two framed links L_1, L_2 are equiv.
iff. D_{L_2} can be obtained from D_{L_1} by fin. seq. of.



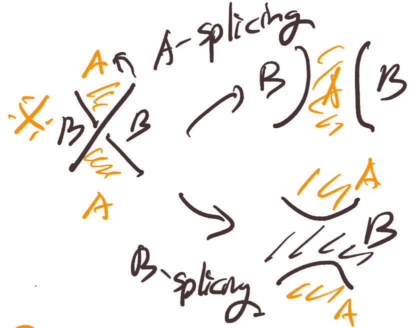


Kauffman bracket

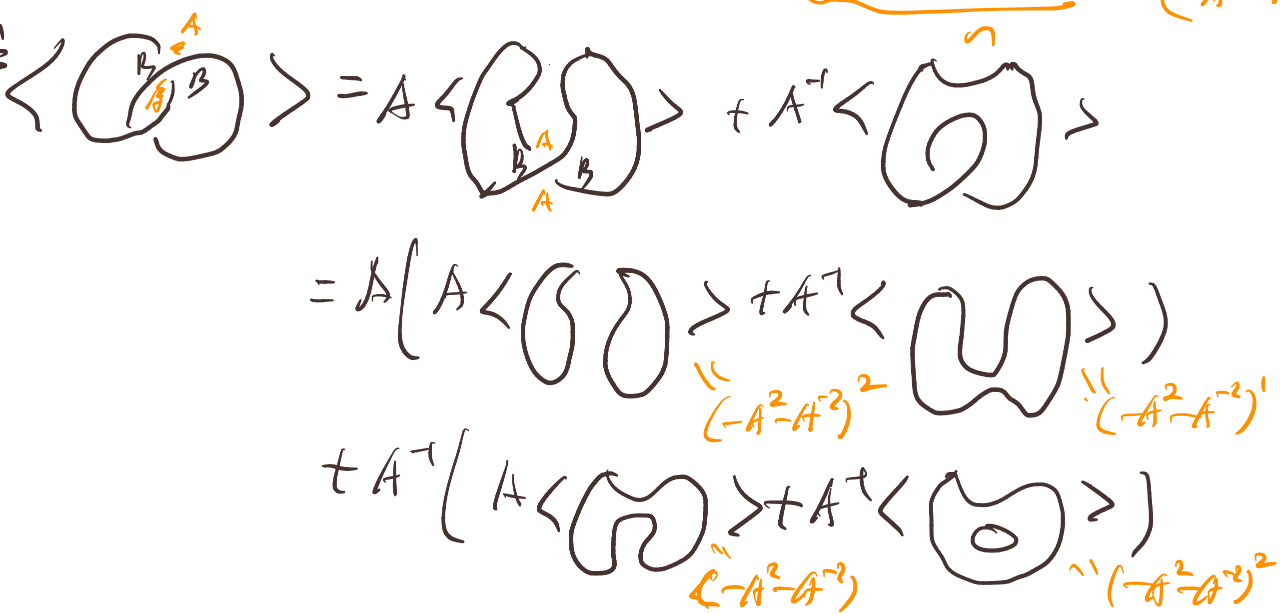
$\langle\ \rangle' = \dfrac{\langle\ \rangle}{(-A^2-A^{-2})}$ $\quad \langle O \rangle = 1$

$\langle\ \rangle : \{\text{diagrams}\} \to \mathbb{Z}[A^{\pm 1}]$ defined by

- $\langle \times \rangle = A\langle)\,(\rangle + A^{-1}\langle \asymp \rangle$
- $\langle L \sqcup O \rangle = (-A^2-A^{-2})L$ $\Rightarrow$ $\langle \underbrace{O, O, \dots, O}_{n} \rangle = (-A^2-A^{-2})^n$



Exa



Thm $\langle D\rangle$ is invariant under RII, RIII, but not RMI

$(\because)$ $\left\langle\right.$  $\left.\right\rangle = (-A^3)^{\pm 1}\langle D\rangle$ ← w.r.t the sign of kink.

Coro $\langle D\rangle$ is invariant for framed links

Def For an oriented knot diag. D.

$$wr(D) = \sum_c \mathrm{sgn}(c) \text{ : the writhe of } D.$$

Thm $(-A)^{-3wr(D)}\langle D\rangle$: invariant under RMs :

$$!!$$

$J(K, A)$ ← the Jones polynomial.

Rem $V_0(t) = J(K, t^{-\frac{1}{4}})$: the Jones polynomial.

Generalizations of Jones polynomial

- 2-variable Homflypt, Kauffman 2-variable

$$a\langle \times \rangle - a^{-1}\langle \times \rangle = z\langle)(\rangle \qquad a\langle \times \rangle - a^{-1}\langle \times \rangle = z\left(\langle)(\rangle - \langle \asymp \rangle\right)$$

Not correct !!!

- Representation theory + quantum group $U_q(sl_2)$

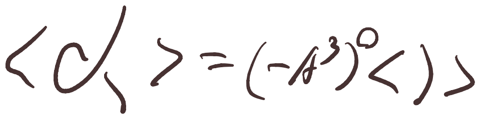
$\Rightarrow$ Reshetikhin–Turaev, colored Jones,

$\vdots$

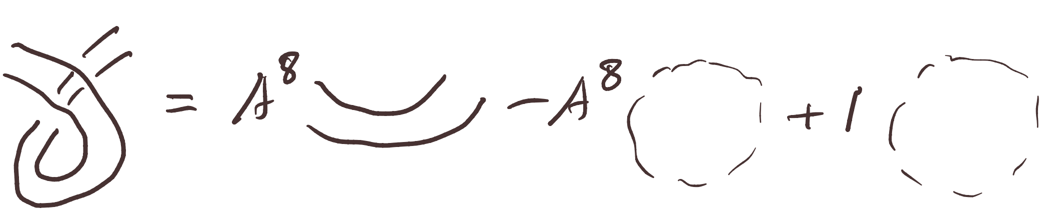
Colored Jones polynomial

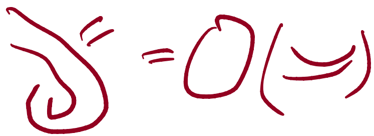
Goal: To generalize Jones polynomial by modifying diagrams.

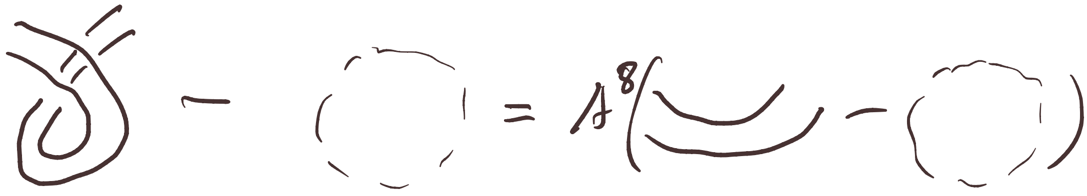
$D^{(1)}$ $D^{(2)}$ $D^{(3)}$



$\Rightarrow \langle D^{(n)} \rangle$ is invariant under RM2, 3, but not RMI.







Take $D^{S_2} = D^{(2)} - 1\phi \Rightarrow D^{S_2} = (A^8)^0 D^{S_2}$

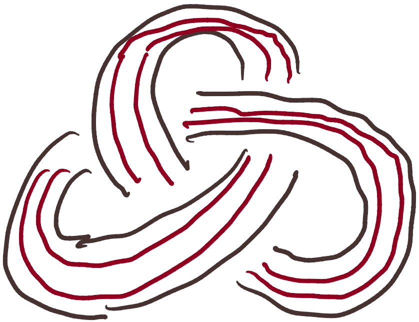
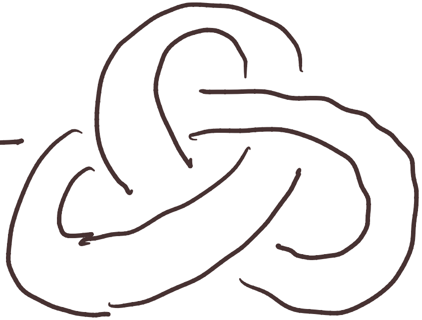
Thm

$(A^8)^{-wr(D)} \langle D^{S_2} \rangle$: an invariant under RMs

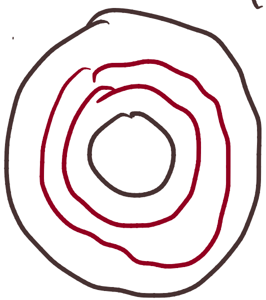
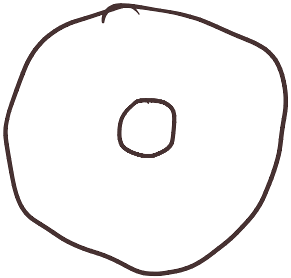
$\|$

$J_2(K, A)$: 2nd colored Jones polynomial.

For $D^{(k)}$, $k \geq 3$, what is D^{S_k} to obtain a knot invariant?

 $-$ 

$D^{(2)} \quad - \quad 1 \cdot \emptyset \quad =: D^{S_2}$

 $-$  $= S^2$

Such a linear combination of links on annulus provides a lin. comb. of links by putting them on the band for the given diagram.

&

S_2 is "eigenvector" of the operator

$\Rightarrow$ Let us consider a kind of vector space consisting of links on 

Kauffman bracket skein module

Def M: an oriented 3-mfd. (w/ boundary)

R: a comm. ring w/ identity (usually $\mathbb{Z}[A^{\pm 1}]$, $\mathbb{Q}$, $\mathbb{C}$)

$\mathcal{L}^{fr}$: the set of ambient isotopy classes of framed links in M

$R\mathcal{L}^{fr} :=$ a free module generated by $\mathcal{L}^{fr} \sqcup \{\emptyset\}$ ← empty link.

S: a submodule of $R\mathcal{L}^{fr}$ gen. by

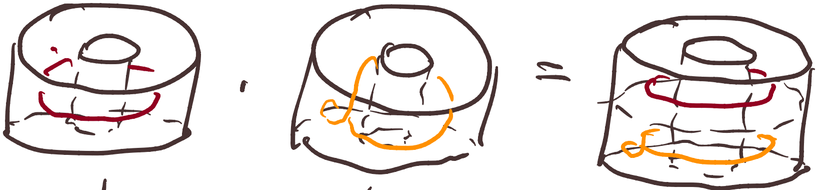
$$\times - A\,)(- A^{-2} \asymp, \quad \bigcirc - (-A^2 - A^{-2})\, \emptyset$$

$\Rightarrow$ $S_{2,\infty}(M, R) := R\mathcal{L}^{fr}/S$: called the Kauffman bracket skein module.

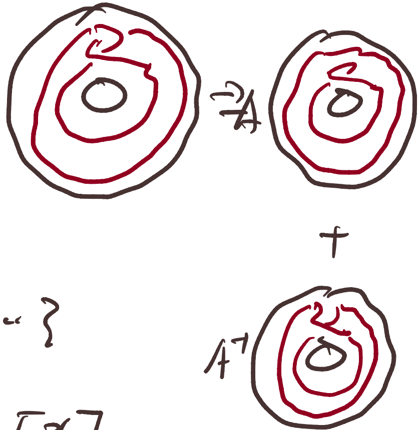
$S_{2,\infty}($  $\times [0,1])$

Kauffman bracket skein algebra of  $\times I$, $S^A($$)$

is $S_{2,\infty}($ $\times I)$ w/ multiplication by stacking links.

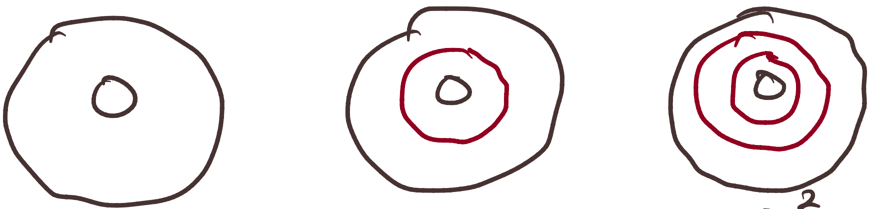


$L_1 \cdot L_2 = L_1 \cdot L_2$



Thm $S^A($$)$: inf. dim. module spanned by $\{1, x, x^2, \ldots\}$

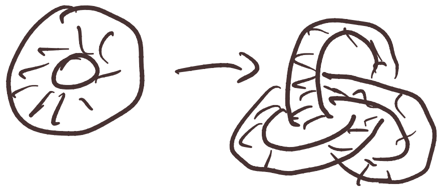
fin. gen. by x, $S^A($$) \cong \mathbb{Z}[A^{\pm 1}][x]$.



1 $\quad x \quad x^2$, ...

Threading operation

D: a diag. of K, $p(x) \in S^A($  $)$

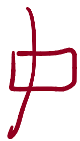
$\Rightarrow D^{p(x)}$ is induced by i: 

If $p(x) = x^n$, D^{x^n}: a n-parallel of D.

If $p(x) = \sum c_i x^i$, $D^{p(x)} = \sum_{i=1}^{n} c_i D^{x^i}$.

Remark $J_2(K, A) = (A^8)^{-wr(D)} \langle D^{x^2-1} \rangle$.

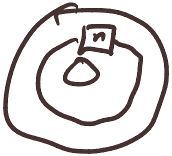
Why D^{x^2-1} worked?

$= A^8$ 

i.e. x^2-1 is an "eigenvector" for R I.

Chebyshev polynomial of 2nd type

$S_0(x)=1$, $S_1(x)=x$, $S_{n+1}(x)=xS_n(x)-S_{n-1}(x)$ in $S^A[$  $]$

Denote it by  $=: S_n(x)$

Lemma  $\leftrightarrow (-1)^n A^{n^2+2n}$ 

Coro $J_n(K,A) = \left((-1)^n A^{n^2+2n}\right)^{-wr(D)} \langle D^{S_n} \rangle$

: invariant under RMs. called n-th colored Jones polynomial.

Exa $J_n(O;A) =$  $^{S_n} = (-1)^n \dfrac{A^{2n+2}-A^{-2n-2}}{A^2-A^{-2}} =: \Delta_n$

<u>Remark</u>

- If you are familiar w/ Temperley-Lieb algebra,

  then  $=: S_n$ is the closure of Jones-Wenzl idempotent.

- The 1st Jones polynomial is original Jones polynomial.

- The colored Jones polynomial is known as quantum invariant.