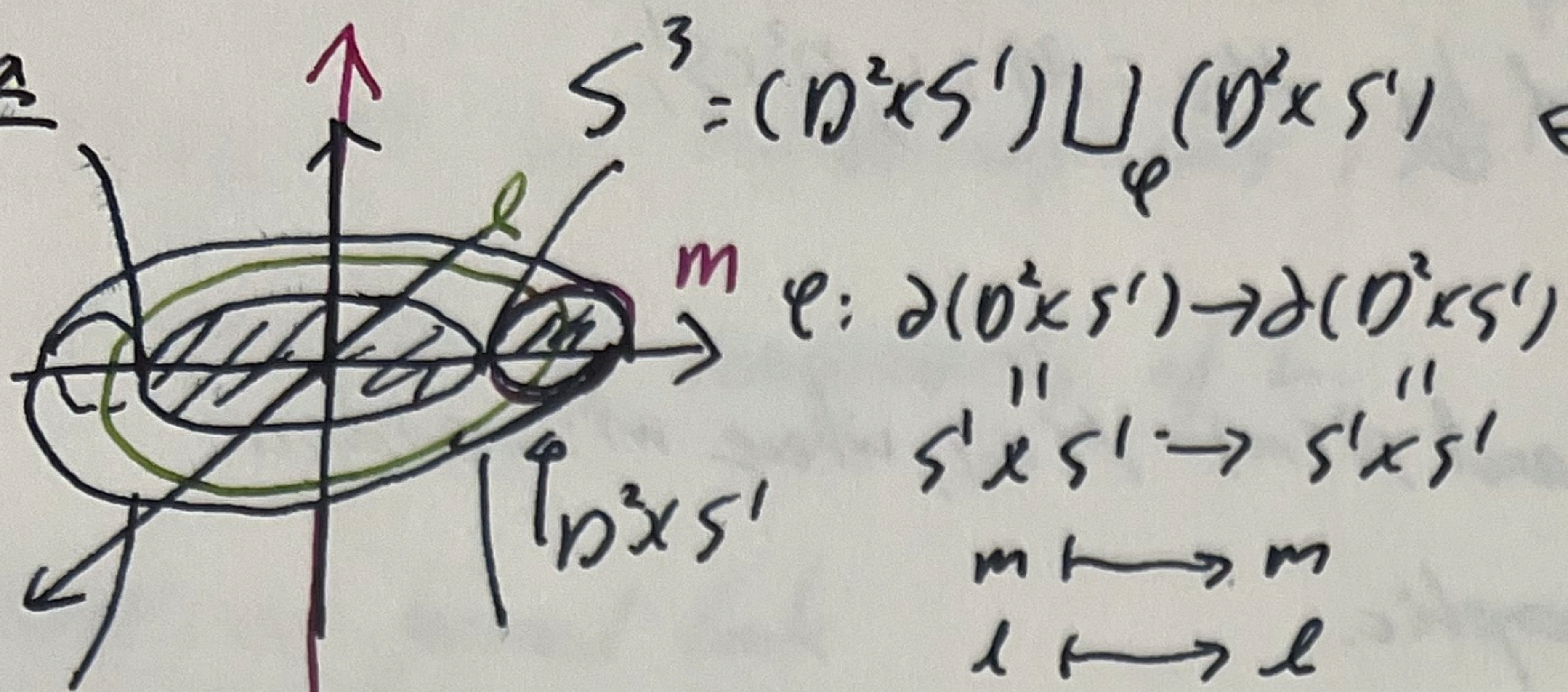


Knot theory and low-dimensional topology

Exa



$$S^3 = (D^2 \times S^1) \cup_{\varphi} (D^2 \times S^1) \iff S^3 = (S^3 - (D^2 \times S^1)) \cup_{\varphi} D^2 \times S^1$$

$$\varphi: \partial(D^2 \times S^1) \rightarrow \partial(D^2 \times S^1)$$

$$S^1 \times S^1 \rightarrow S^1 \times S^1$$

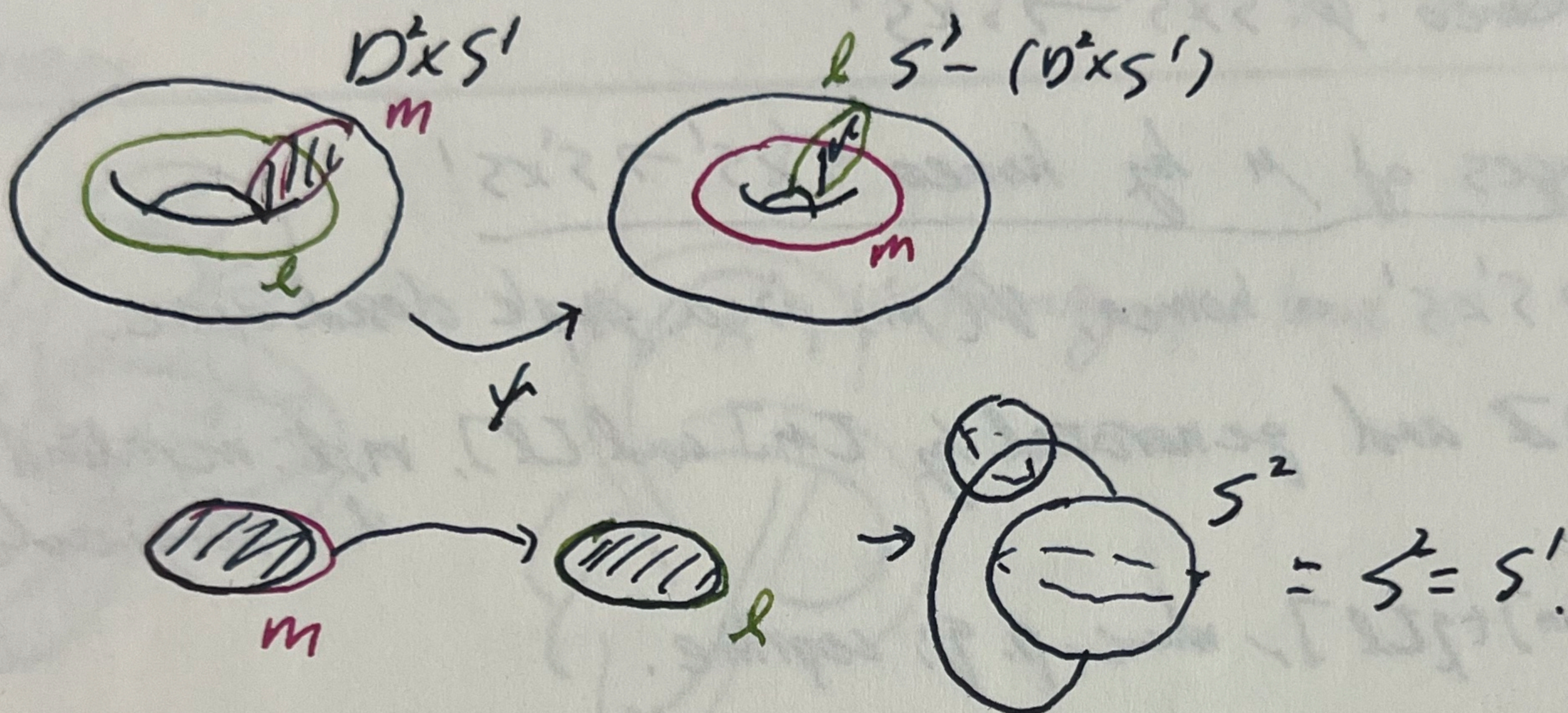
$$m \mapsto m$$

$$l \mapsto l$$

Note that S^3 can be obtained by attaching a solid torus to a torus boundary by a homeomorphism.

This homeomorphism sends m to m and l to l .

Exa Let us consider a homeomorphism $\psi: S^1 \times S^1 \rightarrow S^1 \times S^1$ s.t. $m \mapsto l$ and $l \mapsto m$.



Here, in other words, we can obtain another 3-manifold $S^2 \times S^1$ by removing a solid torus and attaching solid torus again by a homeo.

$$\psi: S^1 \times S^1 \rightarrow S^1 \times S^1$$

Note that, the solid torus in the above examples can be considered as a regular neighborhood of a trivial knot. In general,

Def Let K be a knot. Let $N(K)$ be a regular neighborhood in S^3 .

$$M := (S^3 - N(K)) \cup_{\varphi} (D^2 \times S^1), \quad \varphi: S^1 \times S^1 \rightarrow S^1 \times S^1: \text{a homeo,}$$

is a 3-manifold.

This process is called a Dehn surgery along a knot K on S^3 .

M is called a 3-manifold obtained from S^3 by a Dehn surgery.

Thm $J_n(K, A) = \left(\sum (-1)^n A^{n^2/2n} \right)^{-wr(D)} \langle D^{S_n} \rangle$, n -th colored Jones poly

where S_n : Chebyshev poly in $S(\odot) \cong \mathbb{Z}[A^{\pm 1}][x]$.

Reshetkin-Turaev Invariant

Let D be a framed link diagram of n -components.

Let $\langle \cdot, \dots, \cdot \rangle_D : S(\odot \times I) \times \dots \times S(\odot \times I) \rightarrow S(\mathbb{R}^2)$

the threading operation.

Thm Supp. $A = e^{\frac{1}{4r} 2\pi i}$, i.e. A is $4r$ -th primitive root of unity. $r \geq 3$.

If D' is obtained from D by second Kirby move, then

$$\langle w, \dots, w \rangle_D = \langle w, \dots, w \rangle_{D'}.$$

$$W = \sum_{n=0}^{r-2} \Delta_n S_n(x).$$

Thm Let L, L' be framed links $M_L, M_{L'}$ corresponding 3-mfds.

$M_L \cong M_{L'} : \text{homeo} \iff L, L'$ are equiv. up to

① $\Omega_1, \Omega_2, \Omega_3$

③ Kirby moves.

Revisit to Kauffman bracket and colored Jones polynomial

Kauffman bracket

$\langle \rangle : \{\text{diagrams}\} / \Omega_1, \Omega_2, \Omega_3 \rightarrow \mathbb{Z}[A^{\pm 1}]$

$$\langle \text{Y} \rangle = A \langle \rangle + A^{-1} \langle \rangle$$

$$\langle \bigcirc \cup L \bigcirc \rangle = (-A^2 - A^{-2}) \langle L \rangle.$$

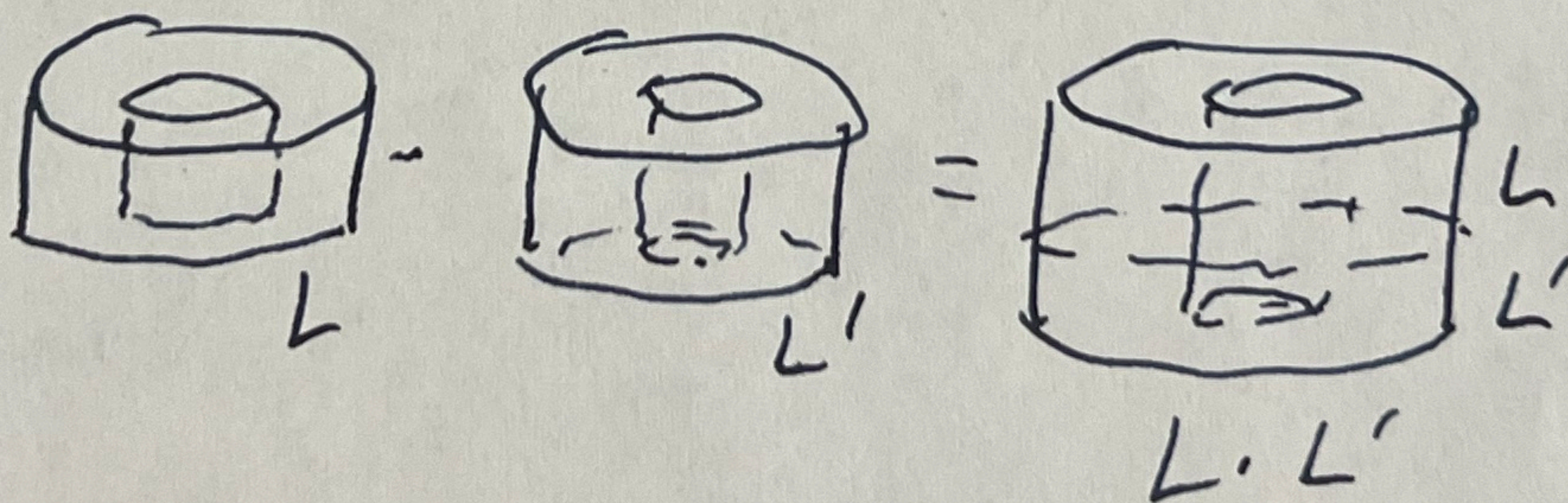
Kauffman bracket skein algebra of $\bigcirc \times I$

$S_{2,0}(\bigcirc \times I) :=$ a module over $\mathbb{Z}[A^{\pm 1}]$ generated by framed links in $\bigcirc \times I$

$\bigcirc \times I$ quotiented by $\text{Y} = A \langle \rangle + A^{-1} \langle \rangle$

$$\bigcirc \cup L = (-A^2 - A^{-2}) L.$$

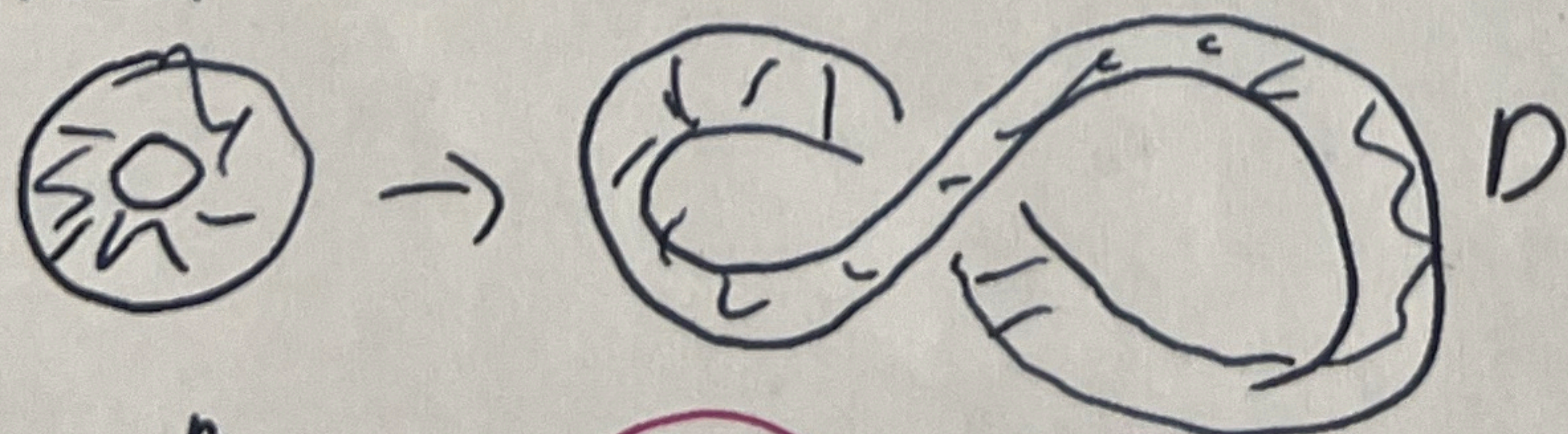
multiplication:



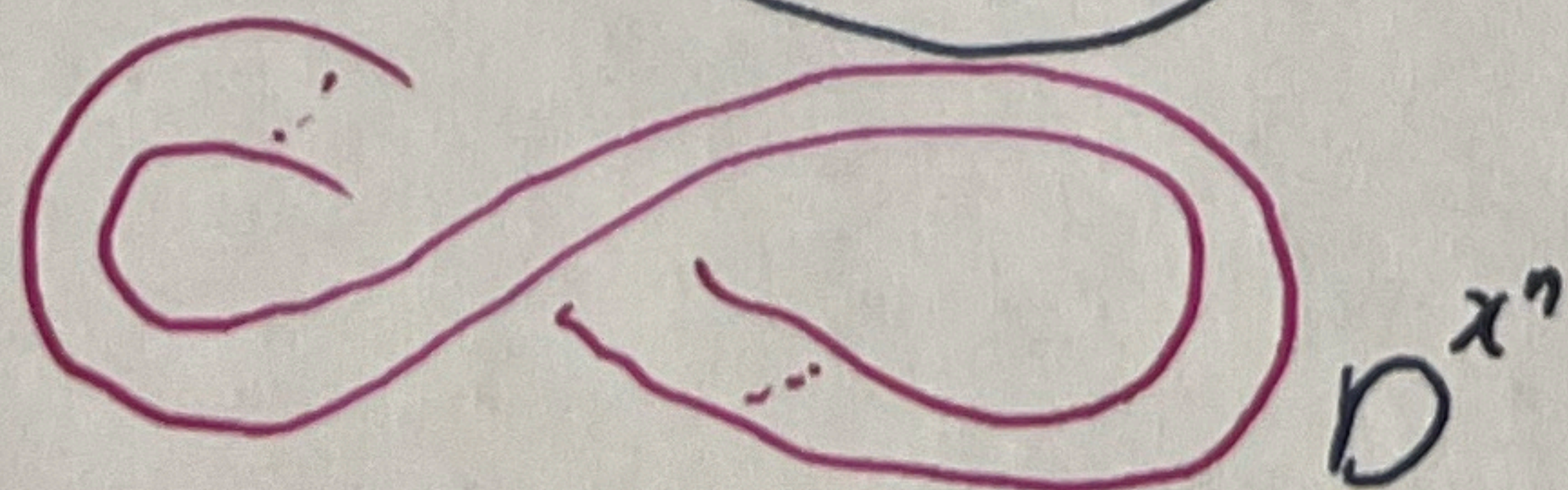
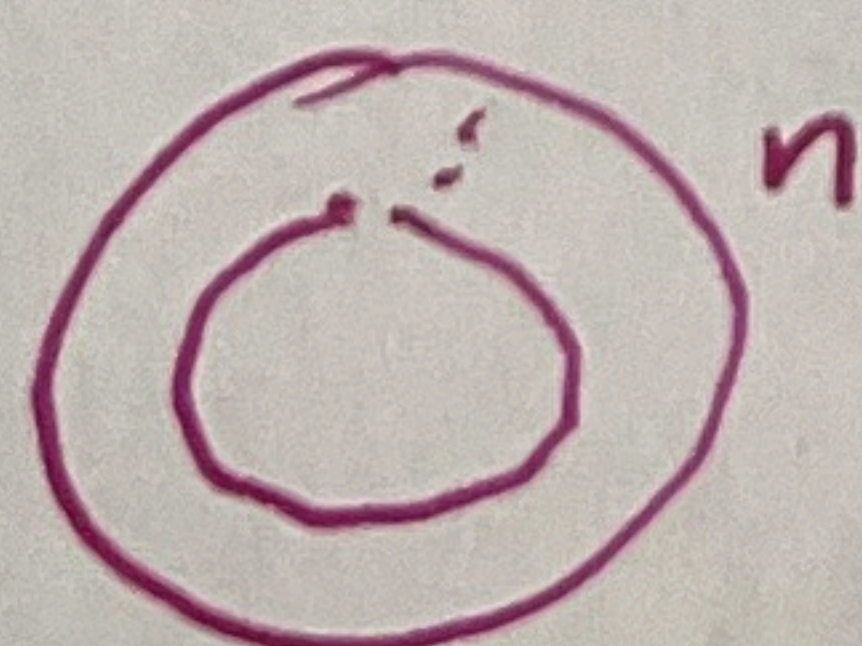
$(S_{2,0}(\bigcirc \times I), \cdot)$ is called a Kauffman bracket skein algebra of $\bigcirc \times I$.

$$\therefore S(\bigcirc) \cong \mathbb{Z}[A^{\pm 1}][x], \quad x := \bigcirc$$

Thready operation



$x^n \mapsto$



$\text{pow} = \sum \zeta_i x_i \mapsto$

$\sum \zeta_i D^{x_i}$

Remed skein algebra of $(\bigcirc) \times \mathbb{Z}$.

S_n : Dehn-Schubert poly.

$$\Delta_n: S_n \hookrightarrow \langle \bigcirc \rangle$$

Rosetta-Turaev invariant

Let $w = \sum_{n=0}^{r-2} \Delta_n S_n(w) \in S(\mathbb{Z} \times S^1)$, $r \geq 3$. D : a framed knot diagram of r -comp.

$\Rightarrow \langle \cdot, \dots, \cdot \rangle_D: S(\bigcirc \times \mathbb{Z})^{\otimes n} \rightarrow S(\mathbb{R}^3)$: thready operation.

Then Suppose that $A = e^{2\pi i \frac{k}{4r}}$, i.e. A is $4r$ th root of unity.

If D' is obtained from D by second Kirby move, then

$$\langle w_1, \dots, w \rangle_D = \langle w_1, \dots, w \rangle_{D'}.$$

For the first Kirby move, we add or delete $(\bigcirc)$ and $(\bigcirc)$.

Let $N(K) = (N_{ij})$ where $N_{ij} = \begin{cases} b(K_i, K_j) & i \neq j \\ w(K_i) & i = j. \end{cases}$

Let $b_+(K)$ and $b_-(K)$ be the number of positive and negative eigenvalues of $N(K)$.

Then let $I(D) = \langle w_1, \dots, w \rangle_D \langle w \rangle_{U_+}^{-b_+} \langle w \rangle_{U_-}^{-b_-}$.

Then it is invariant under Kirby moves and ∂ moves for framed links.

i.e. it is topological invariant for any open closed orientable 3-manifold.

Kirby calculus

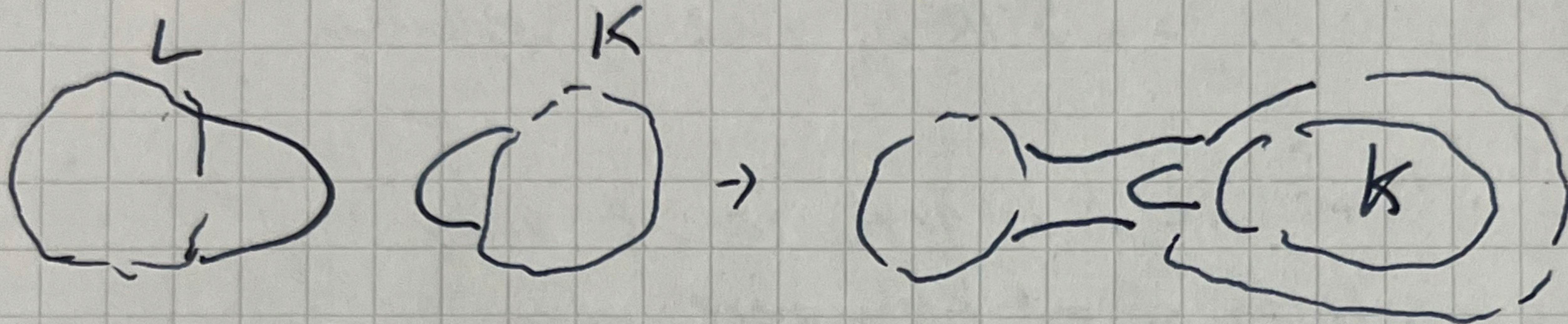
Let L be a framed link.

First Kirby move

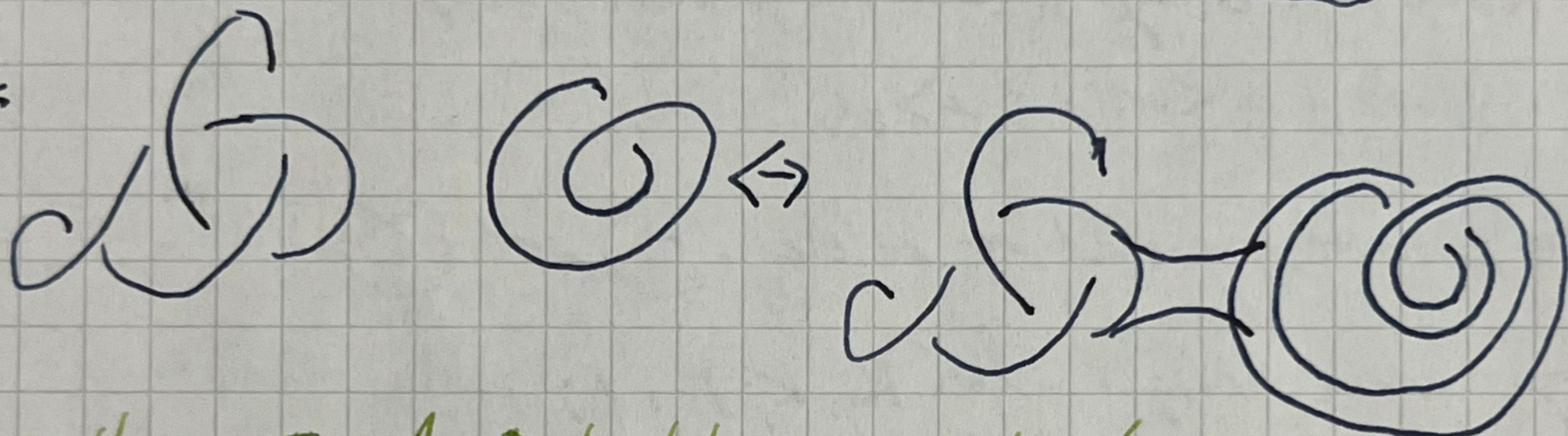
$$L \leftrightarrow L \cup \bigcirc$$

$$\leftrightarrow L \cup \bigcirc$$

Second Kirby move (Handle sliding move)



Exa



Kauffman - Temporarily defined & redefined them. 13.5.12 then?

Thm Let L, L' be framed links. $M_L, M_{L'}$: corresponding 3-mfds.

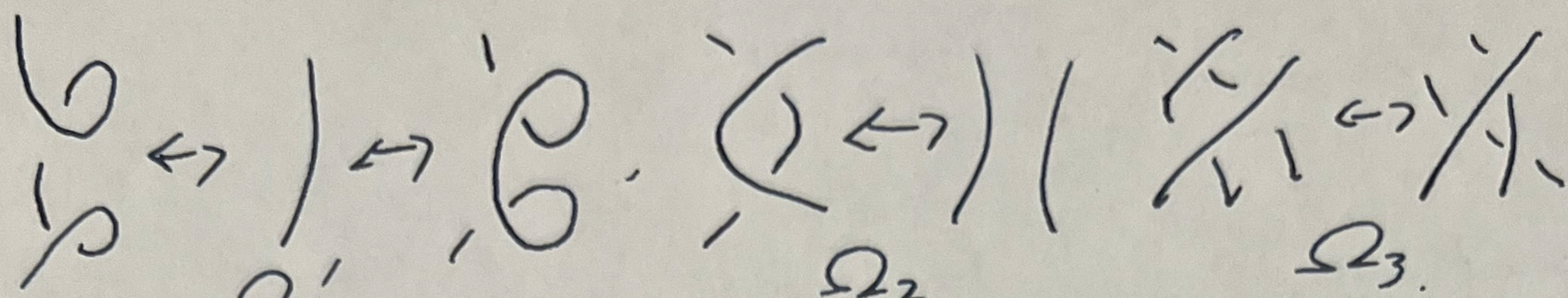
[Kirby] $M_L \cong M_{L'} : \text{homeo} \Leftrightarrow L, L'$ are equivalent up to.

① as framed links

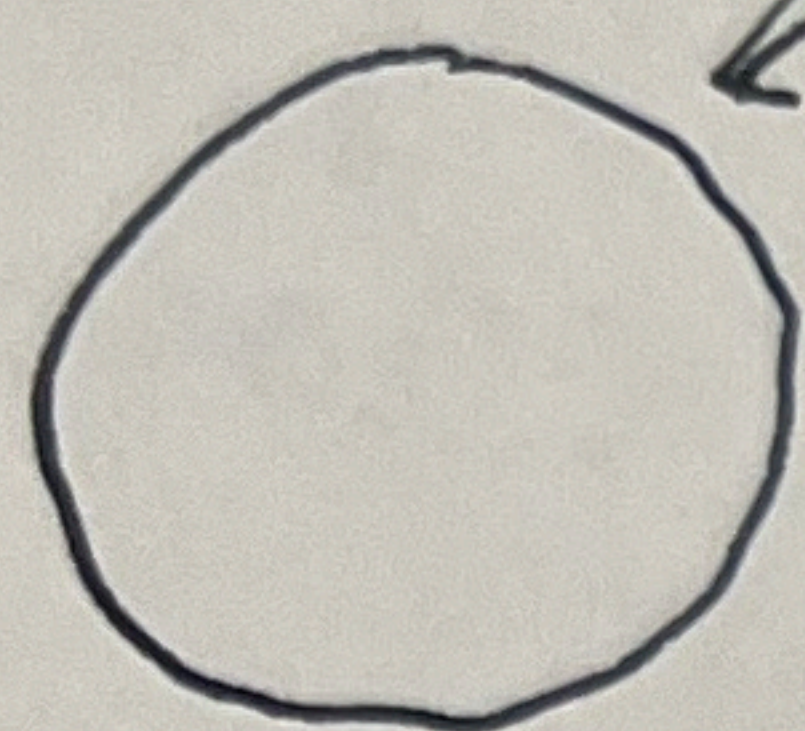
② Kirby moves.

Thm Let L, L' be ^{non.} framed link, $D_L, D_{L'}$ ^{ori.} diagrams

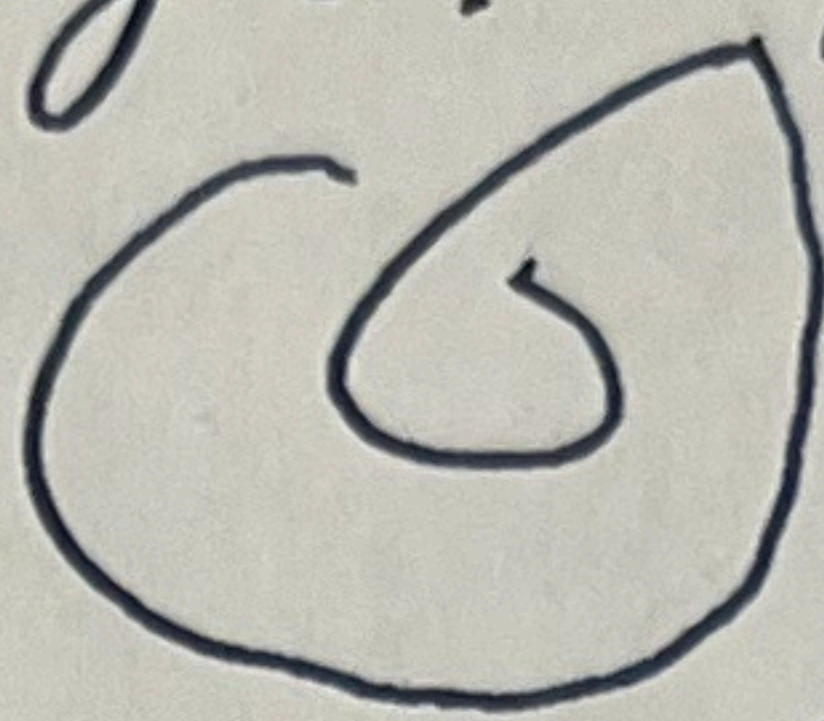
$L \sim L' \Leftrightarrow D_L$ and $D_{L'}$ are equiv. modulo



Exa



$\cong S^2 \times S^1$



$\cong S^3$

(i) $[m] \mapsto 0[m] + 1 \cdot [L]$

$[m] \mapsto 1[m] + 1 \cdot [L]$

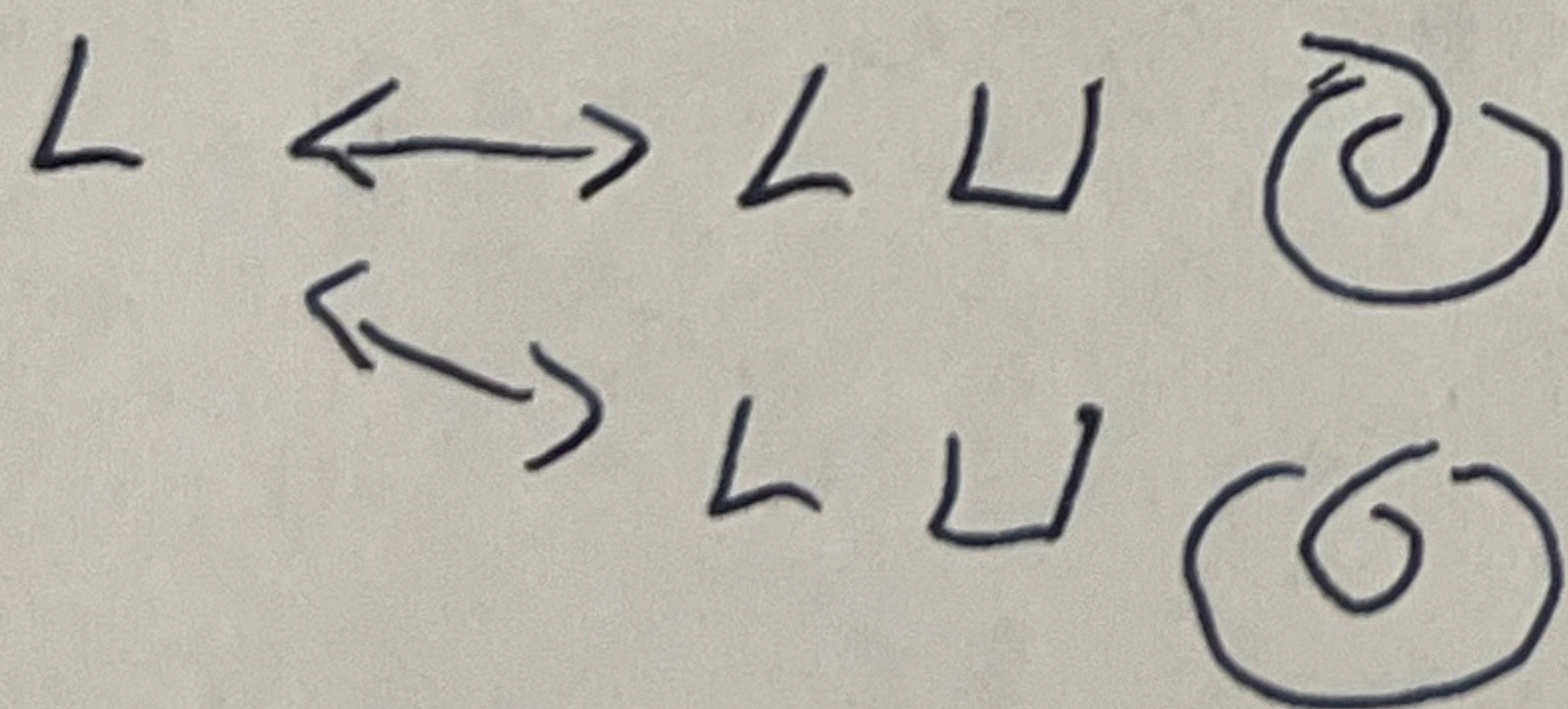
Kirby calculus

$\{\text{framed link diagrams}\} \xrightarrow[\Omega_1, \Omega_2, \Omega_3]{\Omega_1, \Omega_2, \Omega_3} \{\text{3-mfds}\} / \text{homeo.}$

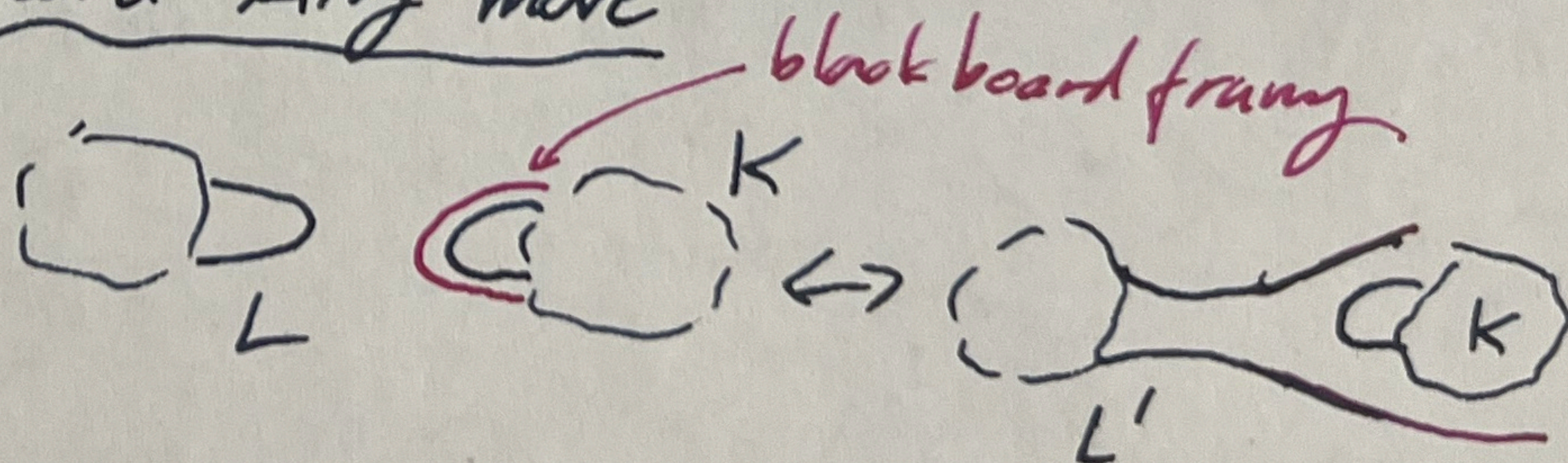
$\{\text{framed link diagrams}\} / \Omega_1, \Omega_2, \Omega_3$

+ Kirby moves. ???

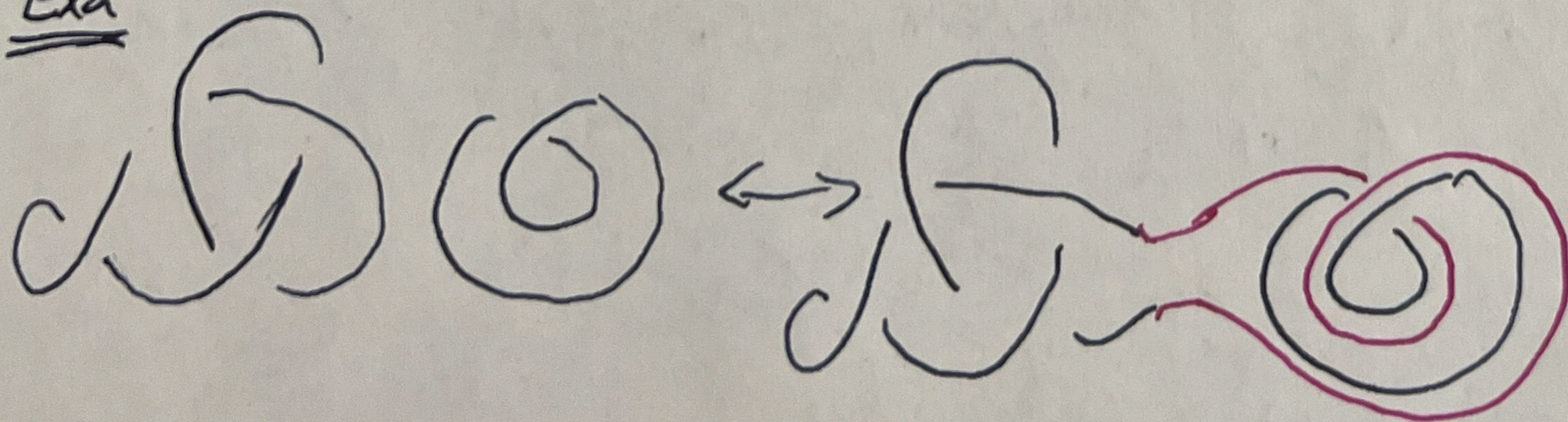
First Kirby move



Second Kirby move



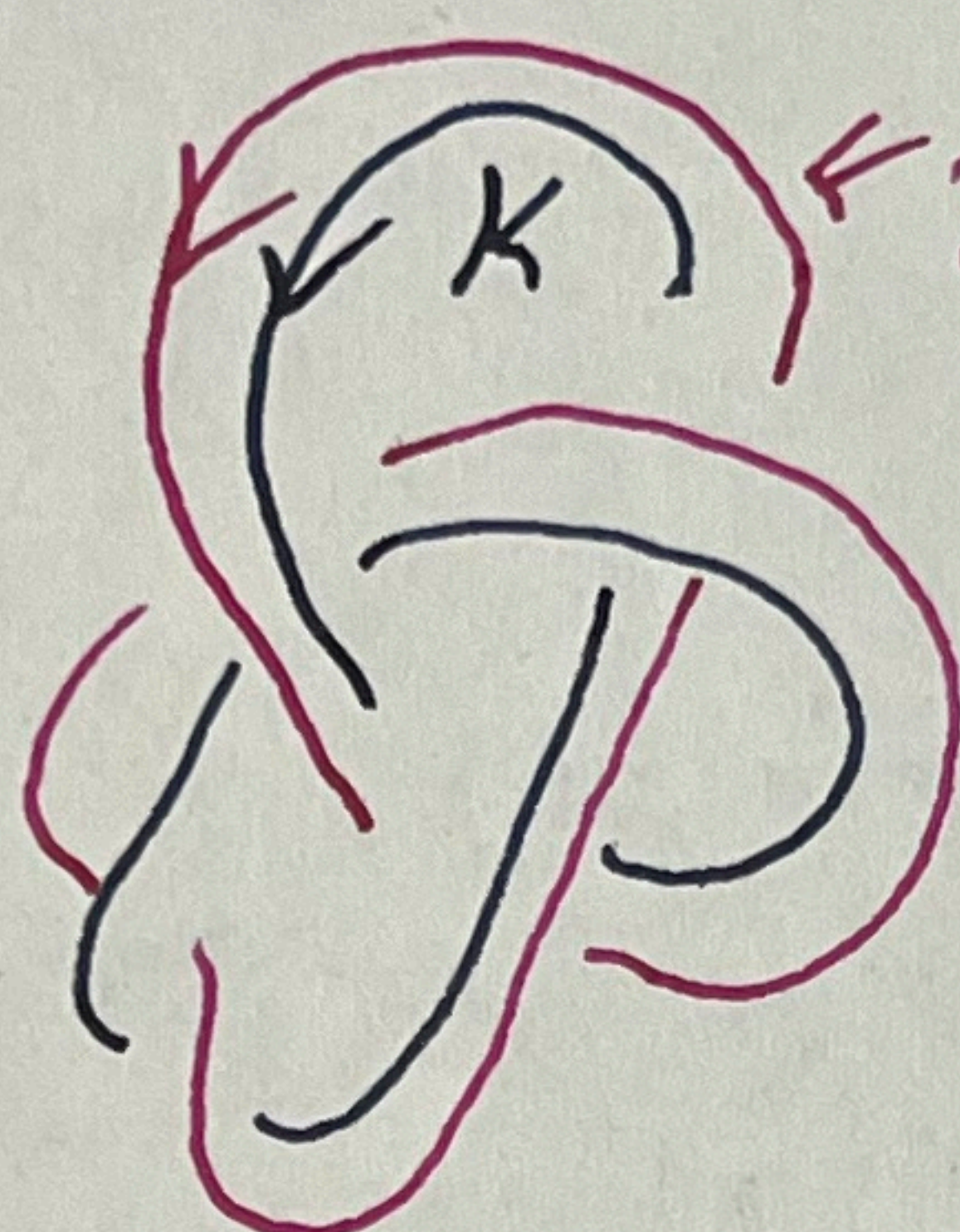
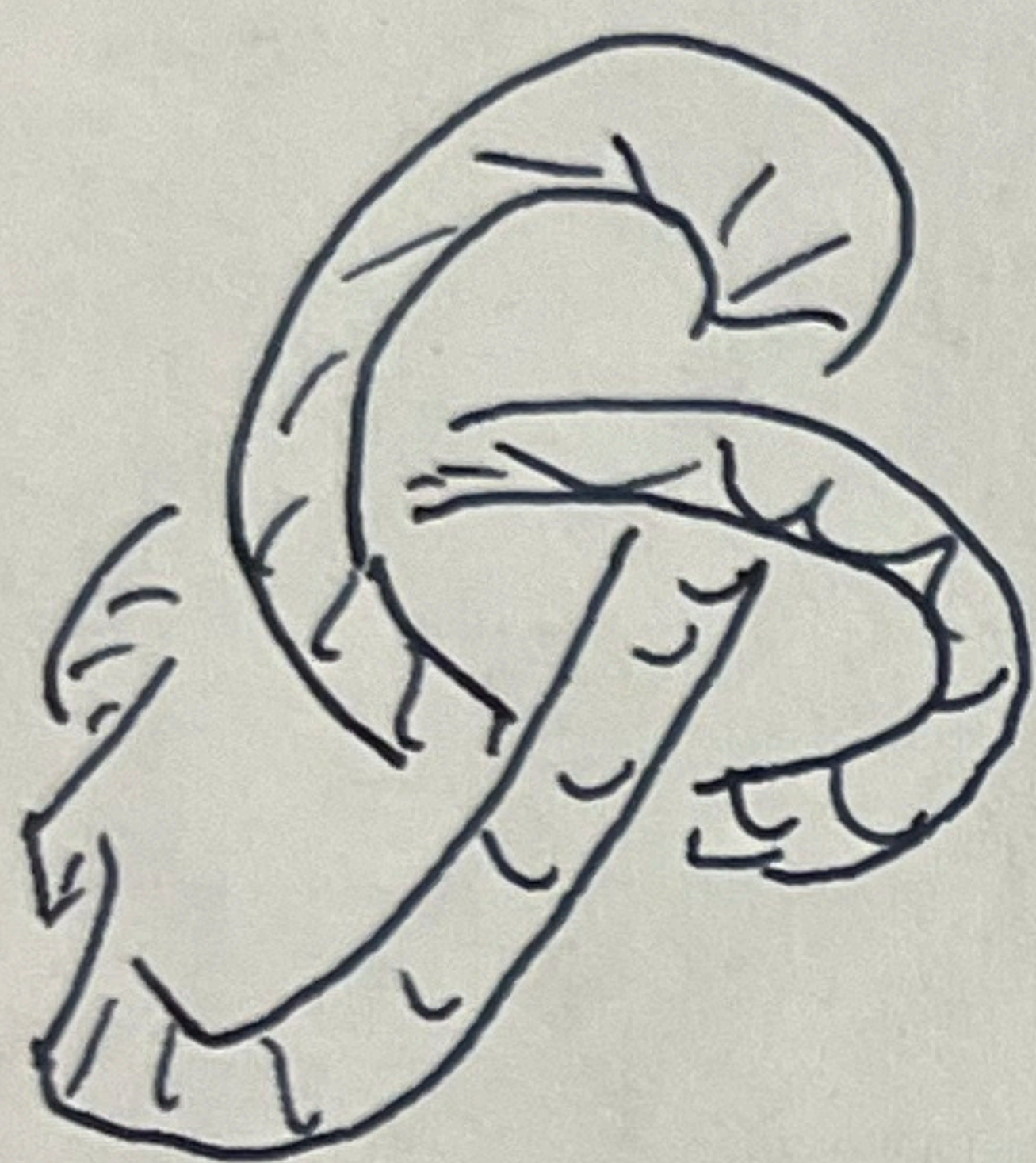
Exa



Coro A framed link L determines a 3-manifold obtained from S^3 by Dehn surgery along L w/ homeomorphisms $S^1 \times S^1 \rightarrow S^1 \times S^1$ which send meridian to framings of components of L .

Revert to framed link

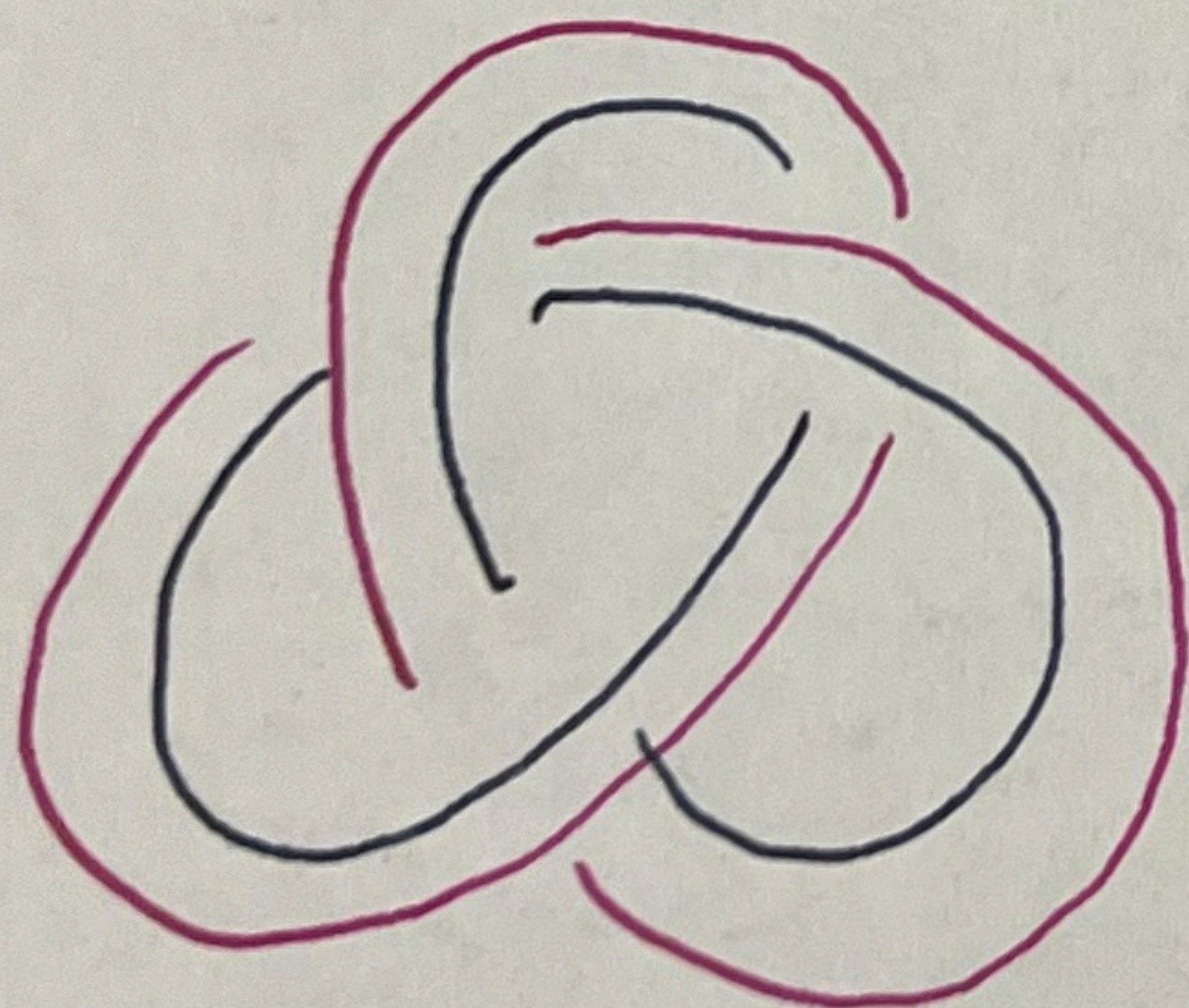
Def A framed knot $(K, \text{framing}) \hookrightarrow S^3$ is a smooth embedding up to ambient isotopy.



← framing of a knot K .

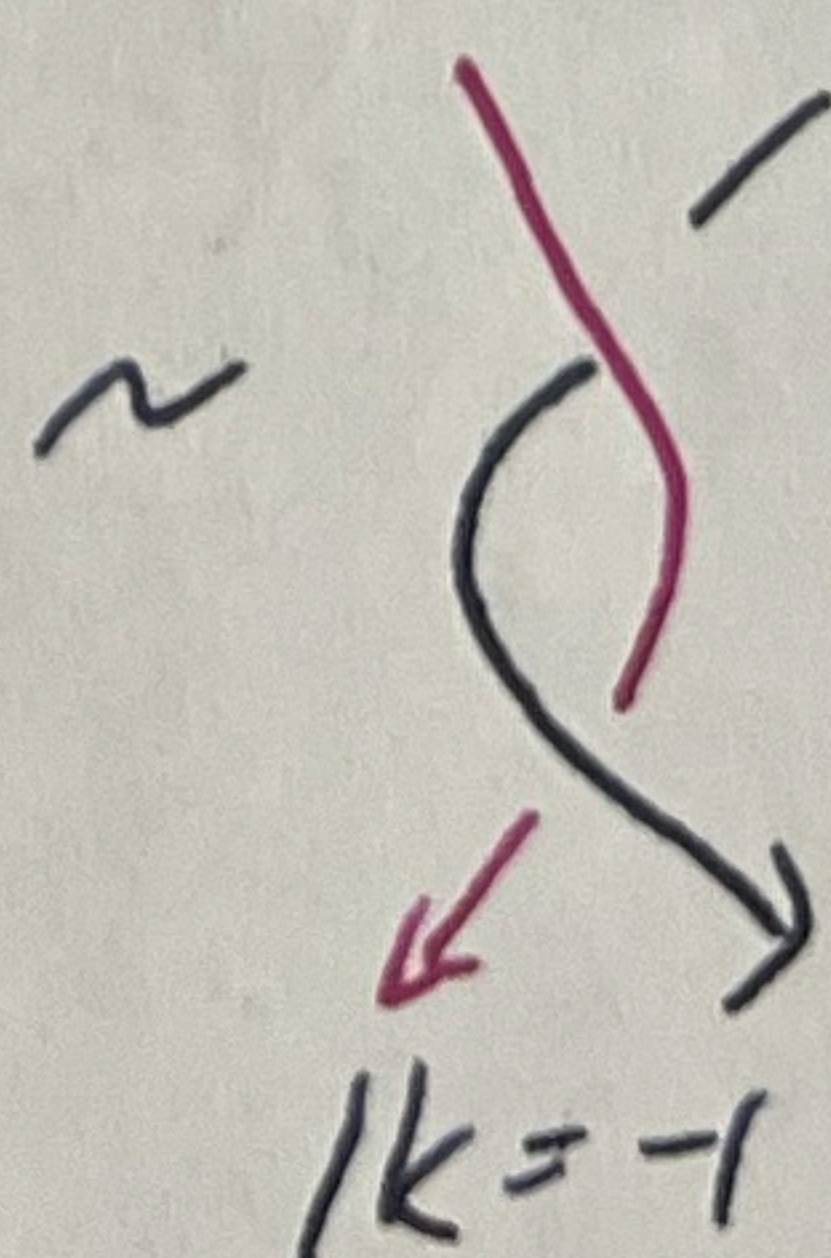
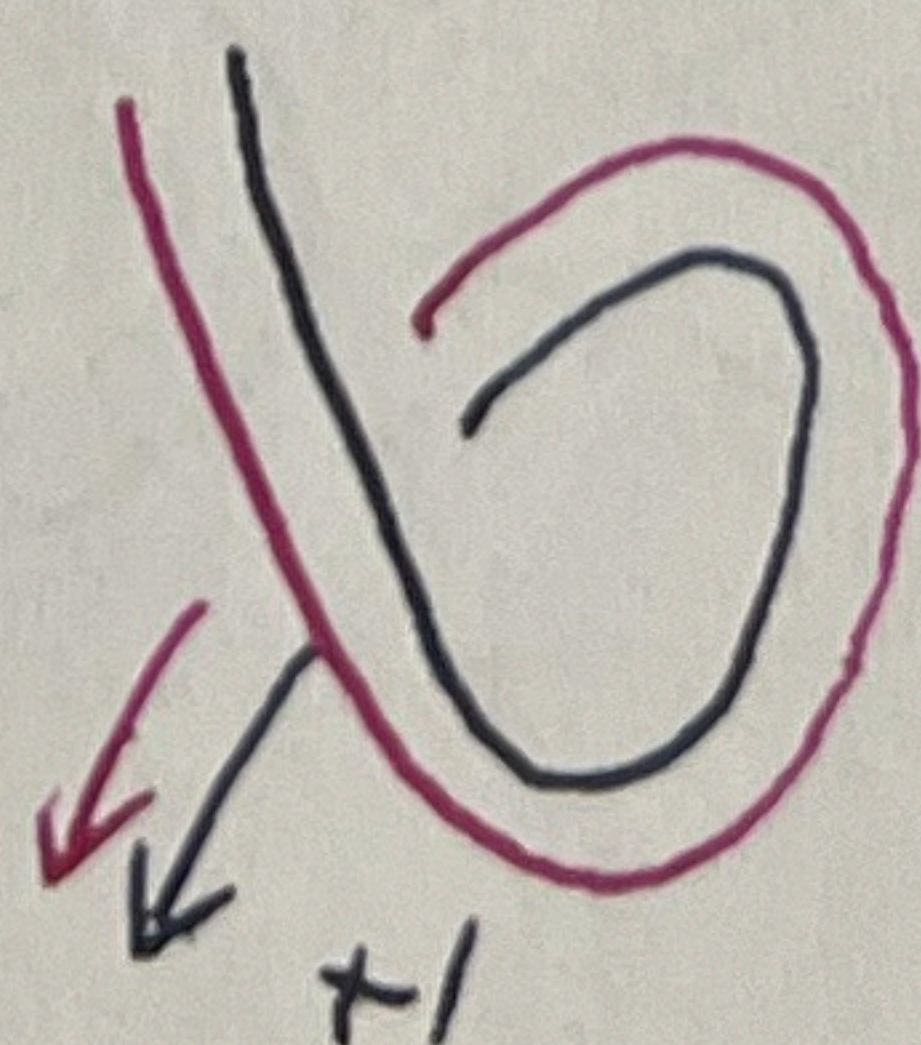
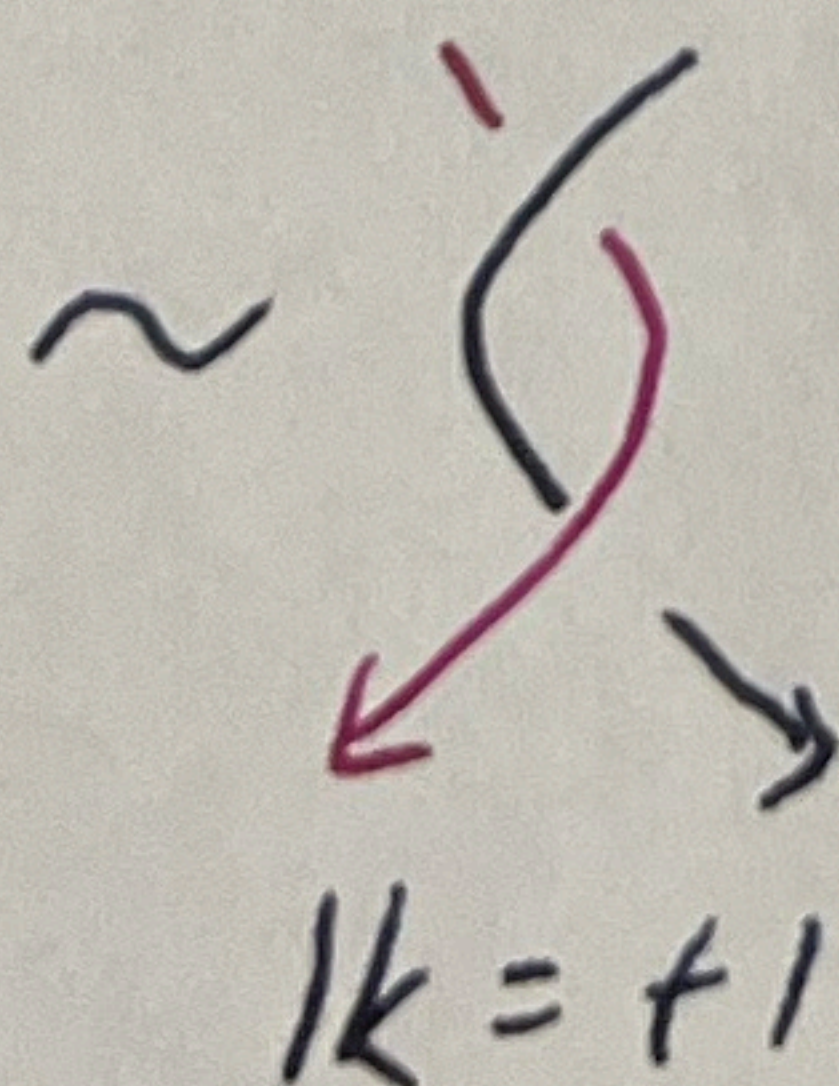
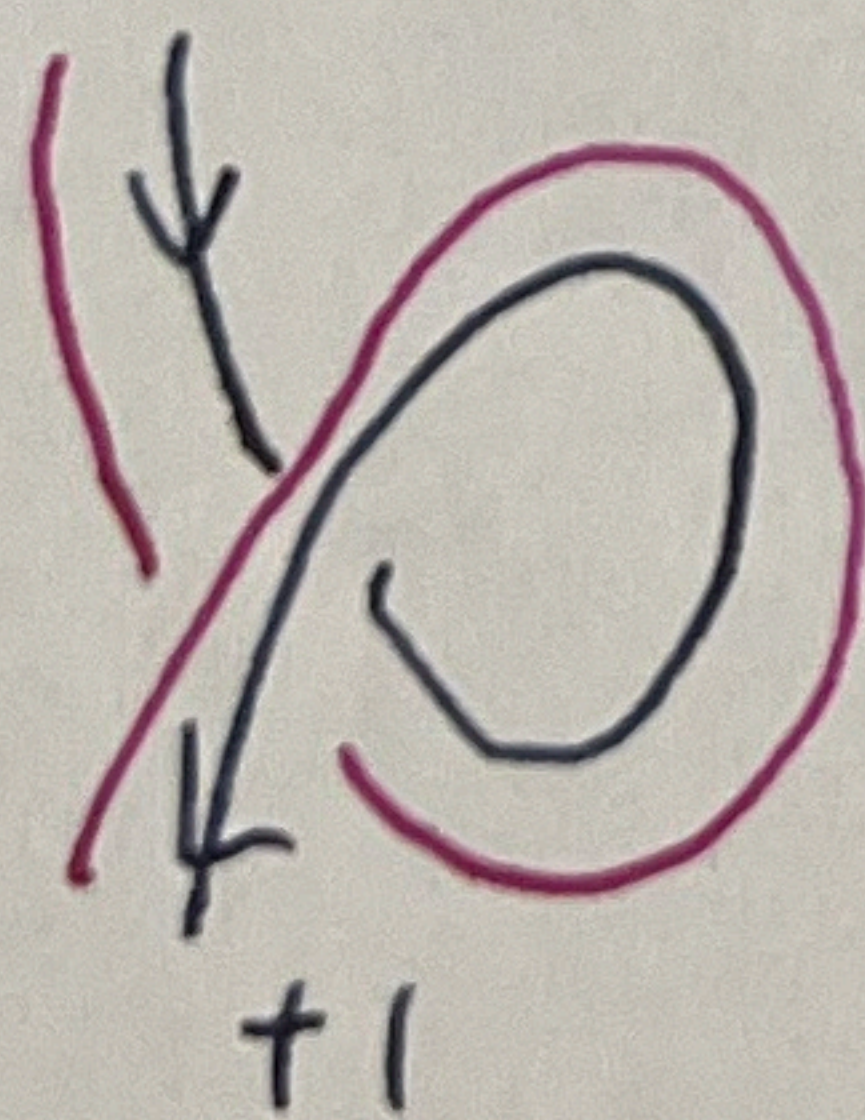
Lemma A framing of K is determined by the linking number between K and K' .

Blackboard framing



The framing of K obtained from blackboard framing of a diagram of K = the writhe of D .

How describe a framing $lk = +1$ by diagram w/ ~~for~~ blackboard framing?



i.e. $\{\text{framed links}\} \leftrightarrow \{\text{diagrams w/ blackboard framing}\}$

Prop Let M be a 3-mfd w/ a torus boundary (exa. $S^2 \times S^1$)

Let M_ϕ be a 3-mfd obtained by $M_\phi = M \cup_\phi D^2 \times S^1$,

$$\phi: S^1 \times S^1 \rightarrow S^1 \times S^1$$

Assume that $\psi: S^1 \times S^1 \rightarrow S^1 \times S^1$ and $\phi(m) = \psi(m)$, where m : meridian.

Then M_ϕ and M_ψ are homeomorphic.

That is, it is enough to determine the image of m to do Dehn surgery along a knot K w/ a homeo: $\phi: S^1 \times S^1 \rightarrow S^1 \times S^1$.

Description of images of m by homeo $S^1 \times S^1 \rightarrow S^1 \times S^1$

For $\phi: S^1 \times S^1 \rightarrow S^1 \times S^1$: a homeo, $\phi(m)$ is a simple closed curve.

$\pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ and generated by $[m]$ and $[l]$, m, l : meridian
 l : longitude

$\Rightarrow \phi([m]) = p[m] + q[l]$, where p, q : coprime.

i.e. image of m by ϕ is determined by rational number $\frac{p}{q}$.

Def. For a knot K in S^3 , a Dehn surgery along K w/ $\phi: S^1 \times S^1 \rightarrow S^1 \times S^1$

$$[m] \mapsto p[m] + q[l]$$

is called a rational surgery along K .

In particular, if $q = \pm 1$, then it is called an integer surgery.

Thm Every closed orientable 3-manifold can be obtained from the sphere S^3 by integer surgery along a link.

Remark α : a simple closed curve s.t. $[\alpha] = p[m] + 1 \cdot [l]$.

