Introduction to spin geometry

1) Dirac bundles and Dirac operators

   $\rightsquigarrow$ Weitzenböck formula

2) Spin structures and spinor bundles

   $\rightsquigarrow$ Schrödinger-Lichnerowicz formula

   $\rightsquigarrow$ Atiyah-Singer index theorem

Clifford algebras

Def

$(V, \langle , \rangle)$ Eucl. vector space

The Clifford algebra of V is defined by

$$Cl(V) := \bigoplus_{r=0}^{\infty} V^{\otimes r} \Big/ v \otimes v + |v|^2,\ v \in V$$

- $Cl(V)$ is an $\mathbb{R}$-algebra with unit $1 \in \mathbb{R} = V^{\otimes 0}$
- $v \in V \Rightarrow v \cdot v = -|v|^2$
- $v \perp w$ in

$$\begin{aligned} -|v|^2 - |w|^2 &= -|v+w|^2 \\ &= (v+w)^2 \\ &= v^2 + vw + wv + w^2 \\ &= -|v|^2 - |w|^2 + vw + wv \end{aligned}$$

$\Leftrightarrow \quad vw = -wv \quad$ in $Cl(V)$

- $Cl_n := Cl(\mathbb{R}^n, \langle , \rangle_{Eucl})$

Examples a) $Cl_1 \cong \mathbb{C}$ (as $\mathbb{R}$-algebras)

$e_1 \longmapsto i$

b) $Cl_2 \cong \mathbb{H} = \text{span}\{1, i, j, k\}$

$e_1 \longmapsto i \quad e_2 \longmapsto j \quad (e_1 \cdot e_2 \longmapsto ij = k)$

Def V_1, V_2 Eucl. vector spaces

Define the $\mathbb{Z}/2$-graded tensor product algebra $Cl(V_1) \tilde{\otimes} Cl(V_2)$ as $Cl(V_1) \otimes Cl(V_2)$ with product

$$(c_1 \otimes d_1) \cdot (c_2 \otimes d_2) = (-1)^{|d_1||c_2|} (c_1 c_2) \otimes (d_1 d_2)$$

Additional explenation that was not mentioned during the talk: the natural grading of the tensor alg. $\bigoplus_{r \geq 0} V^{\otimes r}$ decents to $Cl(V)$ $\rightarrow$ this defines $|.|$ in the equation above, where d_1 and c_2 are assumed to be homogeneous with respect to the grading. The general case is defined by linear extension

Prop $V_1, V_2 \subset V$ orthog. subspaces s.t. $V = V_1 \oplus V_2$

$\Rightarrow$ $\exists$ canonical isom.

$$Cl(V) \cong Cl(V_1) \tilde{\otimes} Cl(V_2)$$

Proof $V_1 \oplus V_2 \longrightarrow Cl(V_1) \tilde{\otimes} Cl(V_2)$ $(v_1, v_2) \longmapsto v_1 \otimes 1 + 1 \otimes v_2$

$\rightarrow f: Cl(V) \longrightarrow Cl(V_1) \tilde{\otimes} Cl(V_2)$

Conversely, $\iota_i : Cl(V_i) \longrightarrow Cl(V)$ $i = 1, 2$

$\rightsquigarrow Cl(V_1) \times Cl(V_2) \longrightarrow Cl(V)$, $(v_1, v_2) \longmapsto \iota_1(v_1) \cdot \iota_2(v_2)$

$\square$

Corollary $n = \dim(V) \Rightarrow \dim Cl(V) = 2^n$

Proof

$V = V_1 \oplus \ldots \oplus V_n$ $\dim(V_k) = 1$

$\Rightarrow Cl(V) \cong Cl_1 \tilde{\otimes} \ldots \tilde{\otimes} Cl_1$

$\Rightarrow \dim(C_1) = \dim(\mathbb{C}) = 2$

$\square$

Prop $\Lambda^* V \cong Cl(V)$ as real vector space

Corollary

$(v_1, \dots, v_n)$ basis of V

$\Rightarrow \quad v_{i_1} \cdots v_{i_k} \qquad i_1 < \dots < i_k \;,\; 0 \le k \le \dim(V)$ basis of $Cl(V)$ as $\mathbb{R}$-vector space

Classification of complex Clifford algebras

$$\mathbb{C}l(V) := Cl(V) \otimes \mathbb{C} \quad , \quad \mathbb{C}l_n := Cl_n \otimes \mathbb{C}$$

Lemma $\quad \mathbb{C}l_1 \cong \mathbb{C} \oplus \mathbb{C} \quad , \quad \mathbb{C}l_2 \cong Mat(2, \mathbb{C})$ as $\mathbb{C}$-alg.

Proof • $\mathbb{C} \oplus \mathbb{C} \longrightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = Cl_1 \otimes \mathbb{C}$

$$(1,0) \longmapsto \tfrac{1}{2}(1 \otimes 1 + i \otimes i)$$
$$(0,1) \longmapsto \tfrac{1}{2}(1 \otimes 1 - i \otimes i)$$

• $Mat(2, \mathbb{C}) = \operatorname{span}\{ \mathbb{1} , \underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\text{square to } -1 \text{ and anticommute}} \}$

$$= \mathbb{H} \otimes \mathbb{C} = \mathbb{C}l_2$$

$\square$

Prop $\forall n \geq 1 \;\; \exists$ canonical isom. of $\mathbb{C}$-alg.

$$\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2$$

__Proof__ $h_1, h_2 \in \mathbb{C}l_2$, $f_1, \dots, f_n \in \mathbb{C}l_n$, $e_1, \dots, e_{n+2} \in \mathbb{C}l_{n+2}$ generators corresp. to standard basis of $\mathbb{R}^l$, $l = 2, n, n+2$

$$\begin{aligned} \mathbb{C}l_{n+2} &\longrightarrow \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2 \\ e_1 &\longmapsto 1 \otimes h_1 \\ e_2 &\longmapsto 1 \otimes h_2 \\ e_k &\longmapsto i f_k \otimes h_1 \cdot h_2 \end{aligned}$$

$\square$

__Corollary__ $\mathbb{C}l_{2k} \cong Mat(2^k, \mathbb{C})$

$\mathbb{C}l_{2k+1} \cong Mat(2^k, \mathbb{C}) \oplus Mat(2^k, \mathbb{C})$

Clifford modules

Def A (complex) Clifford module for $Cl(V)$, or $Cl(V)$-module is a finite dimensional complex vector space W together with a map of $\mathbb{R}$-alg.

$$\rho : Cl(V) \longrightarrow \mathrm{End}_{\mathbb{C}}(W)$$

$$\left(v \cdot w = \rho(v)(w) \right)$$

Let (M,g) be a Riem. mfd.

$\rightsquigarrow$ (T_xM, g_x) Eucl. vector space $\forall x \in M$

$\rightsquigarrow$ get vector bundle $Cl(TM) \longrightarrow M$ with

$Cl(TM)_x = Cl(T_xM)$

There is a vector bundle map

$$Cl(TM) \otimes Cl(TM) \longrightarrow Cl(TM)$$

which restricts to Clifford mult. in each fibre

Dirac bundles and Dirac operators

(M,g) Riem. mfd.

Def • A <u>$Cl(TM)$-module bundle</u> is a Hermitian vector bundle $S \to M$ with a vector bundle map

$$\rho : Cl(TM) \longrightarrow \mathrm{End}_{\mathbb{C}}(S)$$

s.t. ρ restricts to Clifford module $\rho_x : Cl(T_xM) \longrightarrow \mathrm{End}_{\mathbb{C}}(S_x)$

• A <u>Dirac bundle</u> is a Clifford module bundle $S \to M$ with a connection ∇^S on S which is compatible with the Herm. str. $\left(d\langle v,w\rangle_S = \langle \nabla^S v, w\rangle_S + \langle v, \nabla^S w\rangle_S \right)$ and satisfies the Leibniz rule

$$\nabla^S_X (v \cdot u) = \left(\nabla^{TM}_X v\right) \cdot u + v \cdot \nabla^S_X u$$

$\forall\, X, v \in C^\infty(TM)$, $u \in C^\infty(S)$

• $S \to M$ Dirac bundle. The <u>Dirac operator</u> is the composition

$$D : C^\infty(S) \xrightarrow{\nabla^S} C^\infty(T^*M \otimes S) \xrightarrow{g} C^\infty(TM \otimes S) \xrightarrow{c} C^\infty(S)$$

cliff. mult.

If $(e_1, \dots, e_n)$ is an ON-frame of TM, then

$$Du = \sum_{i=1}^{n} e_i \cdot \nabla^S_{e_i} u$$

- $E \to M$ vector bundle with connection ∇

  The <u>second covariant derivative</u> of $u \in C^\infty(E)$ is

$$\nabla^2_{X,Y} u := \nabla_X \nabla_Y u - \nabla_{\nabla_X Y} u$$

- The <u>connection Laplacian</u> $\Delta : C^\infty(E) \longrightarrow C^\infty(E)$ is

$$\Delta := -\sum_{i=1}^{n} \nabla^2_{e_i, e_i}$$

for $(e_1, \dots, e_n)$ an ON-frame of TM

<u>Thm</u> (Weitzenböck formula)

$S \to M$ Dirac bundle

$$\Rightarrow \quad D^2 = \Delta + \mathcal{K}$$

where

$$\mathcal{K} = \sum_{i<j} e_i \cdot e_j \cdot \underset{\substack{\uparrow \\ \text{curv. op. } K^S \in C^\infty(\Lambda^2 T^*M \otimes End(S)) \\ \| \\ \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}}}{K^S(e_i, e_j)} \in C^\infty(End(S))$$

<u>Proof</u> $(e_1, \dots, e_n)$ ON-frame of TM which is synchronous at $x \in M$, i.e.

$$\nabla e_i (x) = 0 \qquad \forall i \qquad \underline{(\#)}$$

Let $u \in C^\infty(S)$. At x we compute

$$D^2 u = \sum_{i,j} e_j \cdot \nabla^s_{e_j} (e_i \cdot \nabla^s_{e_i} u)$$

$$\overset{(*)}{=} \sum_{i,j} e_j \cdot e_i \, \nabla^s_{e_j} \nabla^s_{e_i} u$$

$e_i^2 = -|e_i|^2$
$= -1$

$$= - \sum_i \nabla^2_{e_i, e_i} u + \sum_{j<i} e_j e_i \underbrace{(\nabla_{e_j} \nabla_{e_i} - \nabla_{e_i} \nabla_{e_j})}_{= K^s(e_j, e_i)} u$$

$$[e_i, e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i \overset{(*)}{=} 0$$

$\square$