Introduction to spin geometry 2

Recap

Clifford algebra: $Cl(\underbrace{V, \langle , \rangle}_{\text{Eucl. vector space}}) := \bigoplus_{r \geq 0} V^{\otimes r} \Big/ v \otimes v + |v|^2,\ v \in V$

- $v \in V \Rightarrow v \cdot v = -|v|^2$
- $v \perp w \Rightarrow v \cdot w = -wv$
- $\dim V = n \Rightarrow \dim Cl(V) = 2^n$

$Cl_n := Cl(\mathbb{R}^n, \langle , \rangle_{\text{Eucl.}})$, $\mathbb{C}l_n := Cl_n \otimes_{\mathbb{R}} \mathbb{C}$

Classification of cplx. Clifford algebras

$\mathbb{C}l_{2k} \cong Mat(2^k, \mathbb{C})$, $\mathbb{C}l_{2k+1} \cong Mat(2^k, \mathbb{C}) \oplus Mat(2^k, \mathbb{C})$

Dirac bundle:

Hermitian vector bundle $S \to M$ with

- Clifford module $\rho : Cl(TM) \to End_{\mathbb{C}}(S)$
- metric connection ∇^S on S s.t.

$$\nabla^S_X (v \cdot u) = (\nabla^{TM}_X v) \cdot u + v \cdot \nabla^S_X u$$

$\forall X, v \in C^\infty(TM)$, $u \in C^\infty(S)$

Dirac operator: $D = \sum_{i=1}^n e_i \cdot \nabla^S_{e_i}$

Weitzenböck formula: $D^2 = \Delta + \mathcal{K}$

Spin groups

Def: $Cl_n^\times \subset Cl_n$ mult. group of invertible elements
(each $0 \neq v \in \mathbb{R}^n$ lies in $Cl_n^\times$)

$Spin(n) := \{v_1 \cdot \ldots \cdot v_{2k} \in Cl_n^\times \mid v_1, \ldots, v_{2k} \in \mathbb{R}^n, |v_i| = 1, \underline{k \geq 0}\}$

$v \in \mathbb{R}^n$, $|v| = 1$

- $x = \lambda v$, $\lambda \in \mathbb{R}$ $\Rightarrow$ $-v \cdot x \cdot \underbrace{v^{-1}}_{= -v} = -x$

- $x \perp v$ $\Rightarrow$ $-v \cdot x \cdot v^{-1} = x$

$\Rightarrow$ $\Phi_v : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto v x v^{-1}$ is negative of reflection at $v^\perp \subset \mathbb{R}^n$

$\rightarrow$ extends to group homomorphism

$\Phi : Spin(n) \longrightarrow SO(n)$, $v \mapsto (x \mapsto \Phi_v(x) = v \cdot x \cdot v^{-1})$

called <u>adjoint representation</u> of $Spin(n)$ on $\mathbb{R}^n$

Facts:

- $\ker(\Phi) = \{\pm 1\}$

  $\rightarrow$ $Spin(n) = \{v_1 \cdot \ldots \cdot v_{2k} \mid 0 \leq 2k \leq n, |v_i| = 1\}$

- $Spin(n)$ is a compact Lie-group

- Φ is two-fold covering map.
  In particular, $\dim Spin(n) = \frac{n(n-1)}{2}$

- $Spin(n)$ is connected for $n \geq 2$ and simply connected for $n \geq 3$

$$1 \to \mathbb{Z}/2 \to Spin(n) \to SO(n) \to 1$$

Example: • $Spin(1) = \mathbb{Z}/2$, $Spin(1) \xrightarrow{\Phi} SO(1) = \{1\}$

• $Spin(2) = \{ v_1 \cdot v_2 \mid v_1, v_2 \in S^1 \subset \mathbb{R}^2 \} \subset Cl(2) \cong \mathbb{H}$

$e_1 \mapsto i$, $e_2 \mapsto j$

$(e_1 e_2 \mapsto k)$

Let $v_1 = \alpha i + \beta j \in S^1 \subset span(i,j)$ $\qquad |\alpha|^2 + |\beta|^2 = 1$

$v_2 = a i + b j \in S^1$

$$\to v_1 \cdot v_2 = \underbrace{(-\alpha a - \beta b)}_{=\gamma} + \underbrace{(\alpha b - \beta a)}_{=\eta} k$$

$$\gamma^2 + \eta^2 = 1$$

$\rightsquigarrow Spin(2) = S^1 \subset span(1,k)$

let $\gamma + \eta k \in S^1$, $xi + yj \in \mathbb{R}^2$

$\rightsquigarrow \Phi_{\gamma + \eta k}(xi + yj) = (\gamma + \eta k)^2 (xi + yj)$

$\Rightarrow \Phi : S^1 \to S^1$, $z \mapsto z^2$

From Cliffor modules to Clifford bundle

W Cl_n-module s.t. Clifford mult. with unit vectors in $\mathbb{R}^n$ is an isometry, i.e. $|e| = 1$

$$\langle e \cdot u, e \cdot v \rangle_W = \langle u, v \rangle_W$$

$u, v \in W$

Goal: construct $Cl(TM)$-module bundle $S \to M$ with typical fibre isometric to W

$$\left(S = \begin{array}{c} Cl(TM) \\ \shortparallel \\ \Lambda^* TM \end{array} \right)$$

(M, g) oriented Riem. mfd.

Choose trivialization covering $\Psi_i : TM|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{R}^n$

s.t.

- $\Psi_{ji} = \Psi_j \circ \Psi_i^{-1} : U_i \cap U_j \longrightarrow SO(n)$
- the intersection of finitely many U_i is empty or contractible

Need to find transition maps

$$\rho_{ji} : U_i \cap U_j \longrightarrow Aut(W)$$

s.t.

- coaycle condition

  $\forall ijk$: $\underline{\rho_{ki} = \rho_{kj} \circ \rho_{ji}}$ on $U_i \cap U_j \cap U_k$
- Compatible with Clifford mult.

  $\forall ij$, $\forall x \in U_i \cap U_j \cap U_k$, $\forall v \in \mathbb{R}^n$ and $\forall w \in W$

$$(\Psi_{ji})_x (v) \cdot (\rho_{ji})_x (w) = (\rho_{ji})_x (v \cdot w)$$

For $x \in U_i \cap U_j$ let $\xi \in Spin(n)$ s.t. $\Phi(\xi) = (\Psi_{ji})_x \in SO(n)$

set $(\rho_{ji})_x (w) := \xi \cdot w$

$$\Rightarrow \quad (\psi_{ji})_x(v)\cdot(\rho_{ji})_x(w) = (\xi\cdot v\cdot \xi^{-1})\cdot(\xi\cdot w) = \xi\cdot v\cdot w = (\rho_{ji})_x(v\cdot w)$$

$$\begin{tikzcd}
 & Spin(n) \subset Cl_n \arrow[r] \arrow[d, "\Phi"] & Aut(W) \\
U_i\cap U_j \arrow[r, "\psi_{ji}"'] \arrow[ur, dashed, "\exists \rho_{ji}"] & SO(n) &
\end{tikzcd}$$

Set $\quad z_{ijk} = \rho_{ki}^{-1}\rho_{kj}\rho_{ji} : U_i\cap U_j\cap U_k \longrightarrow Spin(n)$

$\Phi(z_{ijk}) = 1 \quad$ as ψ_{ji} satisfy cocycle cond.

$\rightarrow$ $\mathrm{im}(z_{ijk}) \subset \mathbb{Z}/2 = \ker(\Phi)$

$\rightsquigarrow$ (z_{ijk}) defines cochain $z \in \check{C}^2(M,\mathbb{Z}/2)$

$\rightsquigarrow$ $dz = 0$

$\rightsquigarrow$ $\omega_2(M) = [z] \in H^2(M,\mathbb{Z}/2)$ $\quad$ <u>2^{nd} Stiefel-Witney class</u>

If $\omega_2(M) = 0$, then $\exists \tau \in C^1(M,\mathbb{Z}/2)$ s.t. $d\tau = z$

$\tilde{\rho}_{ji} := \rho_{ji}\tau_{ji}$

$\rightsquigarrow$ $\tilde{\rho}_{ji}$ satisfy cocycle cond.

<u>Def</u> • M <u>spinable</u> if $\omega_2(M) = 0$ $\quad$ (M orient $\Leftrightarrow$ $\omega_1(M) = 0$)

• A <u>Spin-structure</u> on Riem. spin. mfd. (M,g) is a collection (U_i, ψ_i, ρ_{ji})

lifts of ψ_{ji}

orthog. orient.-preserv. local trivial. of TM

open cover of M

- (M,g) Riem spin mfd. , W Cl_n-module

  $\Rightarrow$ $S \to M$ constructed above is called spinor bundle associated with W

<u>Connections on spinor bundles</u>

Let

- $TM|_U \overset{(*)}{\cong} U \times \mathbb{R}^n$, $S|_U \cong U \times W$ as above
- $(e_1, -, e_n)$ standard ON-frame of $U \times \mathbb{R}^n$
- $so(n) = \{A \in \mathbb{R}^{n\times n} \mid A = -A^T\}$ Lie alg. of $SO(n)$
- $\omega = (\omega_i^j) \in \Omega^1(U, so(n))$ connection form of ∇^{TM} w.r.t. $(*)$

  i.e.
  $$\nabla_X^{TM} e_i = \omega(X)(e_i) = \sum_{j=1}^n \omega_i^j(X) e_j \in C^\infty(U, \mathbb{R}^n)$$

<u>Def</u>: Let $w \in C^\infty(U, W)$ be a const. section

Define :
$$\nabla_X^S w = \frac{1}{2} \sum_{1 \le i < j \le n} \omega_i^j(X) \cdot e_i \cdot e_j \cdot w \in C^\infty(U, W)$$

Fact: this defines a global connection on S which turns $S \to M$ into a Dirac bundle

∇^S is called <u>spinor connection</u>

<u>Thm</u> (Schrödinger-Lichnerowicz)

$S \to M$ spinor Dirac bundle for some Cl_n-module W

The corresp. Dirac operator satisfies :
$$D^2 = \Delta + \frac{1}{4} \operatorname{scal}_g$$

Twisted spinor bundles:

Let $E \longrightarrow M$ be a Hermitian vector bundle with compatible connection ∇^E

Set $\nabla^{S\otimes E} := \nabla^S \otimes id + id \otimes \nabla^E$

$\leadsto$ $(S\otimes E, \nabla^{S\otimes E})$ is again a Dirac bundle s.t.

$$D_E^2 = \Delta + \frac{1}{4} scal_g + \underbrace{\mathcal{R}^E}_{= \sum_{i<j} e_i \cdot e_j R^E(e_i, e_j)}$$

R^E: curv. op. of (E, ∇^E)