Scalar curvature Rigidity

Recap

(M^n, g) Riem. mfd is spin iff $\omega_1(M) = 0$ and $\omega_2(M) = 0$

$\rightsquigarrow$ For every Cl_n-module W we can construct a $Cl(TM)$-module bundle $S \to M$ s.t.

$$S_x \underset{\text{isometric}}{\cong} W$$

$S \to M$ carries a compatible connection ∇^S induced by ∇^{LC}

$\rightsquigarrow$ $S \to M$ is a Dirac bundle in a canonical way

$E \to M$ Hermitian vector bundle with metric connection ∇^E

$\rightsquigarrow$ $S \otimes E \to M$ with $\nabla^{S\otimes E} := \nabla^S \otimes id + id \otimes \nabla^E$ is again a Dirac bundle

Schrödinger-Lichnerowicz formula

$$D_E^2 = \Delta + \frac{1}{4} scal_g + \mathcal{R}^E$$

$$\left.\begin{array}{l} \mathbb{C}l_{2k} \cong Mat(2^k, \mathbb{C}) \\ \mathbb{C}l_{2k+1} \cong Mat(2^k, \mathbb{C}) \oplus Mat(2^k, \mathbb{C}) \xrightarrow[pr_{1/2}]{} Mat(2^k, \mathbb{C}) \end{array}\right\} \curvearrowright \mathbb{C}^{2^k} =: \Sigma_n$$

$n = 2k$ or $2k+1$

$Cl_{2k} \hookrightarrow \mathbb{C}l_{2k} \longrightarrow Mat(2^k, \mathbb{C})$

Let $\Sigma M \to M$ be the spinor bundle associated with Σ_n

$\rightsquigarrow$ Assoc. Dirac op.

$$D : C^\infty(\Sigma M) \longrightarrow C^\infty(\Sigma M)$$

$E \to M$ Herm. vector bundle with metric connection

$$\rightsquigarrow D_E : C^\infty(\Sigma M \otimes E) \longrightarrow C^\infty(\Sigma M \otimes E)$$

<u>$n = 2k$</u>

<u>Def</u> $(e_1, \dots, e_n)$ oriented ONB of $\mathbb{R}^n$

$$\omega_{\mathbb{C}} := i^k e_1 \cdot \ldots \cdot e_n \in \mathbb{C}l_n$$

is called the <u>complex volume element</u>

- $\omega_{\mathbb{C}}$ invariant under $SO(n)$

  $\rightsquigarrow \omega_{\mathbb{C}} \in C^\infty(\mathbb{C}l(TM))$

- $\omega_{\mathbb{C}}^2 = id_{\Sigma M}$
- $\nabla^S \omega_{\mathbb{C}} = 0$

$\Rightarrow$ orthog. splitting $\Sigma M \cong \Sigma M^+ \oplus \Sigma M^-$

± 1-eigenbundles of $\omega_{\mathbb{C}}$

called <u>chirality decomposition</u> of ΣM

$$\rightsquigarrow D_E^{\pm} : C^\infty(\Sigma M^{\pm} \otimes E) \longrightarrow C^\infty(\Sigma M^{\mp} \otimes E)$$

$$D = \sum_i e_i \cdot \nabla^S_{e_i} \qquad D\omega_{\mathbb{C}} = -\omega_{\mathbb{C}} D$$

$$D_E = \begin{pmatrix} 0 & D_E^- \\ D_E^+ & 0 \end{pmatrix}$$

M closed (compact without boundary), then

- D_E extends to a self-adj. op.

$$\overline{D_E} : H^1(\Sigma M \otimes E) \longrightarrow L^2(\Sigma M \otimes E)$$

- $\overline{D_E}$ is Fredholm

$$\rightsquigarrow \operatorname{index}(\overline{D_E}) = \dim(\ker(\overline{D_E})) - \underbrace{\dim(\operatorname{coker}(\overline{D_E}))}_{=\dim(\ker(\overline{D_E}^*))} = 0$$

$\rightsquigarrow$ take insted

$$\operatorname{index}(\overline{D_E}^+) = \dim(\ker(\overline{D_E}^+)) - \dim(\ker(\overline{D_E}^-))$$

<u>Thm</u> (Atiyah-Singer)

$$\operatorname{index}(\overline{D_E}^+) = \int_M \underbrace{\hat{A}(TM)}_{\in \Omega^{4*}(M)} \wedge \underbrace{ch(E)}_{\in \Omega^{2*}(M,\mathbb{C})} = \langle \hat{A}(TM) \cup ch(E), [M] \rangle$$

Scalar curvature

Def: (M^n, g) Riem. mfd., $p \in M$

$$\frac{\mathrm{Vol}(B_\varepsilon(p) \subset M)}{\mathrm{vol}(B_\varepsilon(0) \subset \mathbb{R}^n)} = 1 - \frac{1}{6(n+2)} \mathrm{scal}_g(p)\, \varepsilon^2 + \mathcal{O}(\varepsilon^3) \qquad \varepsilon \searrow 0$$

Example: $\overset{g_{\mathrm{Eucl.}}}{\mathbb{R}^n}$ $\quad \overset{g_0}{S^n}$ $\quad \overset{g_{\mathbb{H}}}{\mathbb{H}^n}$

scal $\quad 0 \quad n(n-1) \quad -n(n-1)$

Thm (Kazdan-Warner '75)

M closed mfd. $\dim(M) = n \geq 3$, $f \in C^\infty(M)$ s.t. $\exists x \in M : f(x) < 0$

$\Rightarrow \exists$ Riem metric g with $\mathrm{scal}_g = f$

Thm (Lichnerowicz)

M closed spin mfd. s.t. $\hat{A}(M) = \int_M \hat{A}(TM) \neq 0$

$\Rightarrow$ M is not psc

Proof Let $u \in \ker(D)$ $\qquad$ $\mathrm{index}(D) = \hat{A}(M)$

$$(u,v)_{L^2} = \int_M \langle u,v \rangle_x \, d\mathrm{vol}(x)$$

$$\Rightarrow 0 = (D^2 u, u)_{L^2}$$

$$= (\underset{\nabla^*\nabla}{\underset{\uparrow}{\Delta}} u + \tfrac{1}{4} \mathrm{scal}_g u, u)_{L^2}$$

$$= \underbrace{\|\nabla^S u\|^2_{L^2}}_{\geq 0} + \tfrac{1}{4} (\mathrm{scal}_g u, u)_{L^2} > 0 \quad \lightning$$

If $\mathrm{scal}_g > 0 \Rightarrow \lightning$ $\qquad \square$

Thm 1 (Llarull '98)

(M^n, g) closed connected spin mfld. , $n \geq 2$

a) $\mathrm{scal}_g \geq n(n-1) = \mathrm{scal}_{S^n}$

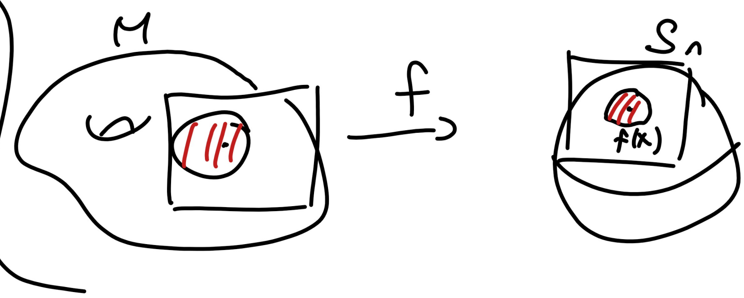
b) $f : (M,g) \longrightarrow (S^n, g_0)$ smooth , $\deg(f) \neq 0$ s.t.

- f 1-Lipschitz $n = 2$ — $|df| \leq 1$
- f area-nonincreasing $n \geq 3$

$\quad$ $\Lambda^2 df : \Lambda^2 TM \longrightarrow \Lambda^2 TS^n$

$\quad$ $|\Lambda^2 df| \leq 1$

$\Rightarrow f$ Riem isometry



special case $M = S^n$, $f = \mathrm{id} : (S^n, g) \longrightarrow (S^n, g_0)$

$\rightsquigarrow g = g_0$

Proof of thm 1 /2

$\left.\begin{array}{l} (\Sigma M, \nabla) \longrightarrow M \\ (\Sigma S^n, \nabla^0) \longrightarrow S^n \end{array}\right\}$ spinor bundles

w.r.t. smooth background metric

$(E, \nabla^E) := f^*(\Sigma S^n, \nabla^0)$ Lipschitz bundle

$\rightarrow D_E : \overset{\text{Lip}}{C^\infty}(\Sigma M \otimes E) \longrightarrow \overset{L^2}{C^\infty}(\Sigma M \otimes E)$

1) Atiyah-Singer

(*) approx. g by smooth metrics
$\rightsquigarrow$ cont. family of of Dirac operators

n even $\qquad \Sigma M \cong \Sigma M^+ \oplus \Sigma M^-$

$$\overline{D_E} \text{ is Fredholm}$$

$$\text{index}(\overline{D_E^+}) \overset{(*)}{=} \langle \hat{A}(TM) \cup ch(E), [M] \rangle$$

$$= \dots = \underbrace{\deg(f)}_{\neq 0} \cdot \underbrace{\chi(S^n)}_{=2} \neq 0$$

$\Rightarrow \exists$ harmonic spinor $u \in \ker(D_E)$

n odd : consider Dirac op. on $(M \times S^1, g + r^2 g_{S^1})$

$\underset{r\to\infty}{\rightsquigarrow}$ harm. spinor of D_E on M

2) Schrödinger-Lichnerowicz

$u \in \ker(D_E) \cancel{\subset} C^\infty(\Sigma M \otimes E)$

$$0 = \|D_E u\|^2_{L^2} = \left(\underline{\overline{D_E}^2} u, u\right)_{L^2}$$

distr. sense

$$= \underbrace{\left((\nabla^* \nabla u, u)_{L^2}\right)}_{= \|\nabla u\|^2_{L^2} \geq 0} + \underbrace{\tfrac{1}{4}(\mathrm{scal}_g u, u)}_{\geq \frac{1}{4} n(n-1) \|u\|^2_{L^2}} + \underbrace{(\mathcal{R}^E u, u)_{L^2}}_{\geq 0}$$

At $\overset{\text{a.e.}}{\text{every}}$ $x \in M$

$$\langle \mathcal{R}^E u, u \rangle = \sum_{i<j} \langle e_i \cdot e_j \cdot R^E(e_i, e_j) u, u \rangle$$

$$= \sum_{i<j} \langle e_i \cdot e_j \, (f^* R^{\Sigma S^n})(e_i, e_j) u, u \rangle$$

$$= \sum_{i<j} \langle e_i \cdot e_j \otimes f_*(e_i) \cdot f_*(e_j) \cdot u, u \rangle$$

Sing. value decomp.:

$\exists (\varepsilon_1, \dots, \varepsilon_n)$ ONB of $T_{f(x)} S^n$ and $\mu_1, \dots, \mu_n \in \mathbb{R}^n$ s.t.

$$f_*(e_i) = \mu_i \varepsilon_i$$

f 1-Lipschitz

$\Rightarrow$ $0 \leq \mu_i \leq 1$ only if $n=2$

$$= \sum_{i<j} \mu_i \mu_j \langle e_i e_j \otimes \varepsilon_i \cdot \varepsilon_j u, u \rangle$$

$$\geq - \sum_{i<j} \underbrace{\mu_i \mu_j}_{\leq 1} |u|^2$$

$$\geq - \frac{1}{4} n(n-1) |u|^2$$

$\Rightarrow$ $\mathrm{Scal}_g \equiv n(n-1)$, $\mu_i \mu_j = 1 \quad \forall i \neq j$

$\Rightarrow$ $\mu_i = 1 \quad \forall i$

$\Rightarrow$ f local isometry a.e.

$\Big($ $\Rightarrow$ $f: (M, g) \longrightarrow (S^n, g_0)$ Riem. covering

$\Rightarrow$ f Riem. isometry $\Big)$

$\square$

df orient. preserv. a.e.

$\Longrightarrow$ f metric isometry

Reshetnyak $\square$

Thm 2 (Cecchini–Hanke–Schick '22, C–H–S–S '25)

M^n smooth closed connected spin mfd., $n \geq 2$ even, odd

a) $g \in W^{1,p}(T^*M \otimes T^*M)$, $p > n$ admissible Riem. metric

s.t. $\mathrm{Scal}_g \overset{\text{distr.}}{\geq} n(n-1)$

b) $f: (M,g) \longrightarrow (S^n, g_0)$ Lipschitz, $\deg(f) \neq 0$ s.t.

- f 1-Lipschitz $n=2$
- f area-nonincreasing a.e. $n \geq 3$

$\Longrightarrow$ $f: (M, d_g) \longrightarrow (S^n, d_0)$ metric isometry