Scalar Curvature Rigidity 2

Thm 1 (Llarull '98)

(M^n, g) closed connected spin, $n \geq 2$

a) $\mathrm{scal}_g \geq n(n-1) = \mathrm{scal}_{g_{S^n}}$

b) $f : (M,g) \to (S^n, g_0)$ smooth, $\deg(f) \neq 0$ s.t.

- f 1-Lipschitz, $n = 2$
- f area-nonincreasing, $n \geq 3$

$\Rightarrow f$ Riem. isometry

- Goette-Semmelmann '02: replace S^n by mfd. with positive Riem. curvature operator and non-vanishing Euler charakteristic

- Lott '21: Goette-Semmelmann for compact mfds. with boundary

- Brendle '24: Convex polytopes

- Bär-Brendle-Hanke-Wang '24: S^n with antipodal points removed n even n odd

Thm 2 (Cecchini-Hanke-Schick '22, C-H-S-S '25)

M^n smooth closed connected spin, $n \geq 2$

a) $g \in W^{1,p}(T^*M \otimes T^*M)$, $p > n$, admissible Riem. metric s.t.

$$\mathrm{scal}_g \overset{\text{distr.}}{\geq} n(n-1)$$

b) $f : (M, g) \longrightarrow (S^n, g_0)$ Lipschitz, $\deg(f) \neq 0$ s.t.

- f 1-Lipschitz , $n = 2$
- f area-nonincreasing a.e. , $n \geq 3$

$\Rightarrow f : (M, d_g) \longrightarrow (S^n, d_0)$ metric isometry

Remark Lee-Tam '22: Thm 2 in the case f is 1-Lipschitz $\forall n \geq 2$

Def M closed mfd.

- $0 < \vartheta \leq 1$, $g_r \in C^\infty((0, \vartheta), C^\infty(T^*M \otimes T^*M))$ family of Riem. metrics
- $g_0 \in C^2(T^*M \otimes T^*M)$

s.t. $g_r \xrightarrow{r \to 0} g_0$ in C^2 and

$$|r\partial_r g_r| \longrightarrow 0 \quad , \quad |r^2\partial_r^2 g_r| \longrightarrow 0 \quad \text{as } r \to 0$$

Then $Cg_r := dr^2 + r^2 g_r$ on $(0,\vartheta)\times M$ is called generalized cone metric.

(W,G) is a smooth Riem. mfd. with cone-like singularities, if

- $\exists\, \underline{K} \subset W$ compact
- $\exists\, 0 < \vartheta \leq 1$ and Riem. isometry

$$\vartheta : (W\setminus K, G|_{W\setminus K}) \longrightarrow ((0,\vartheta)\times\partial K, \underset{\substack{\uparrow \\ \text{generalized} \\ \text{cone metric}}}{Cg_r})$$

<u>Thm 3</u> (C-H-S-S '25)

(W^{n+1}, G) Riem. spin mfd. with cone-like sing.

$n \geq 3$ odd

- $scal_G \geq (n+1)n$
- $f : W \longrightarrow S^{n+1}$ Lipschitz, area-nonincreasing a.e.

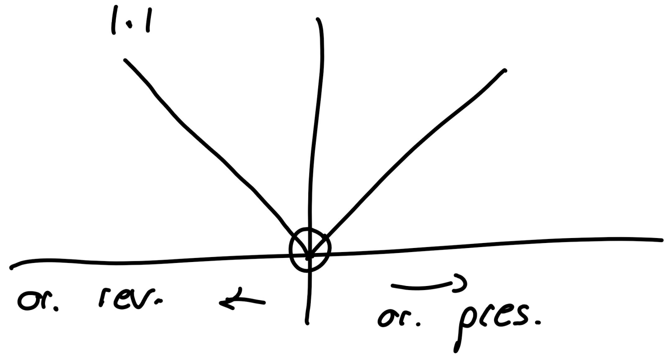
  $\deg(f) \neq 0$

$\Rightarrow$ f Riem. isometry onto a punctured sphere

Proof of Thm 2 part 2

local isometry $\rightsquigarrow$ metric isometry for n even

to show: f either or. pres. or or. rev. a.e.



$\rightsquigarrow$ 1.1 local isometry a.e. but

$$2 = d(-1,1) \neq d(|-1|, |1|) = 0$$

$0 \neq u \in \ker(D_E^+)$ $\qquad\qquad$ $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$

From SL-formula

- $(e_i \cdot e_j \otimes f_* e_i \cdot f_* e_j) \cdot u = u \qquad (1)$
- $\nabla^{S\otimes E} u = 0$ a.e. $\rightsquigarrow$ $|u| = \text{const.}$ a.e. $\qquad (2)$

Let $x \in M$ s.t. $d_x f$ exists

Let $\omega_{\mathbb{C},M}$, $\omega_{\mathbb{C},S^n}$ be the volume forms

$\omega_{\mathbb{C},M} = i^k e_1 \cdot \ldots \cdot e_n$

$$\Rightarrow (\omega_{\mathbb{C},M} \otimes \omega_{\mathbb{C},S^n}) \cdot u = u \qquad (3)$$

f local isometry a.e. $\Rightarrow$ $\omega_{\mathbb{C},S^n} = \pm i^k f_* e_1 \cdot \ldots \cdot f_* e_n$

$+$ if f or. pres.

$-$ — " — or. rev.

$$\Rightarrow \pm u \overset{(3)}{=} i^{2k} (e_1 \cdot \ldots \cdot e_n \otimes f_* e_1 \cdot \ldots \cdot f_* e_n) \cdot u$$

$$\overset{(1)}{=} (-1)^k u$$

$\Rightarrow$ f orient $\begin{cases} \text{pres. in } x \text{ if } k \text{ even} \\ \text{rev. — " — } k \text{ odd} \end{cases}$

$\overset{(2)}{\Rightarrow}$ holds a.e.

$\Rightarrow$ f metric isometry

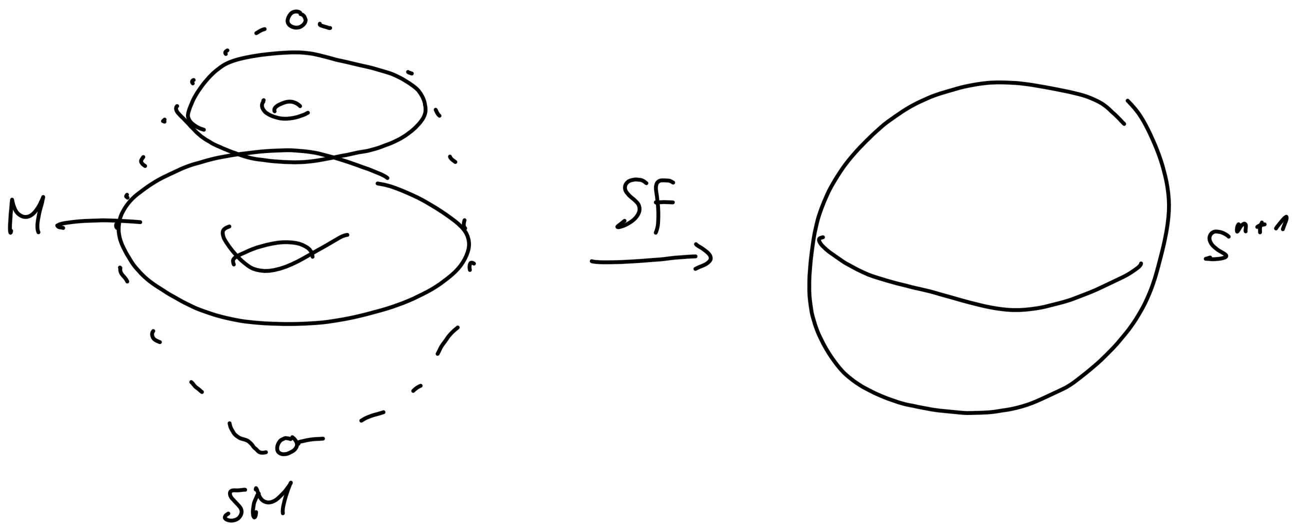
Reshetnyak

<u>n odd</u>

Consider the spherical susp.

$SM := M \times (0, \pi) \qquad Sg := dr^2 + \sin^2(r) g$

$Sf : (SM, Sg) \longrightarrow (SS^n, S_{g_n}) = (S^{n+1} \setminus \{N, S\}, g_{S^{n+1}})$



$\mathrm{scal}_{Sg} \geq (n+1)n = \mathrm{scal}_{g_{S^{n+1}}}$

But f area-nonincr. does not imply that

Sf — " —

Solution: first show f 1-Lipschitz using spectral flow

Then generalize even dim. proof to $SM \xrightarrow{Sf} S^{n+1}$

Caveat: Only have SL-formula for $p > n+1$ on SM

Spectar flow

"Def" Given a "continuous" path of self-adj. Fredholm op. D_t, $t \in [a,b]$, the spectral flow is

$sf(D_t)$ ":=" $\#\{$ eigenvalues passing from $-$ to $+\}$

$- \#\{$ — " — $+$ to $-\}$

Thm (Getzler '93)

M^{2k+1} closed connected spin, D Dirac op. on $\Sigma M \otimes \mathbb{C}^N$ for $N \in \mathbb{N}$

Let $u: M \longrightarrow U(N)$ smooth and

$$D_t := (1-t)D + t\, u^{-1} D u \quad . \quad t \in [0,1]$$

$$\rightsquigarrow \quad sf(D_t) = -\frac{1}{(-2\pi i)^{k+1}} \int_M \hat{A}(TM) \wedge Ch(u)$$

If $M = S^{2k+1}$ $\quad sf(D_t) = \deg(u)$

Def For $u \in C^\infty(\Sigma M)$ set , $X \in TM$

$$\nabla^t_X u = \nabla^{\Sigma M}_X u + t X \cdot u$$

∇^t-parallel spinors are called Killing spinors

Example: $n = 2k$, $n = 2k+1$

For $t = \pm \frac{1}{2}$, ΣS^n can be trivialized by $\nabla^{\pm\frac{1}{2}}$-parallel spinors i.e.

$$\exists\ \Sigma S^n \xrightarrow[\sim]{V_\pm} S^n \times \mathbb{C}^{2^k} \qquad (*)$$

$f: M \longrightarrow S^n$, consider $E = f^*(S^n \times \mathbb{C}^{2^k}) = M \times \mathbb{C}^{2^k}$ with trivial connection

$\rightsquigarrow D_E$ twisted Dirac op.

(*) induce trivializ. $U_\pm : E \xrightarrow{\sim} M \times \mathbb{C}^{2^k}$

Set $U = U_+ U_-^{-1} : M \longrightarrow U(\mathbb{C}^{2^k})$

Li-Su-Wang

$\Rightarrow$ $sf(D_t) = \deg(f) \neq 0$

$\Rightarrow$ for some $t \in [0,1]$ $\exists u \in \ker(D_t)$

Abstract cone operators

Consider $SM = (0,\pi) \times M$, $Sg = dr^2 + \sin^2(r)\, g$

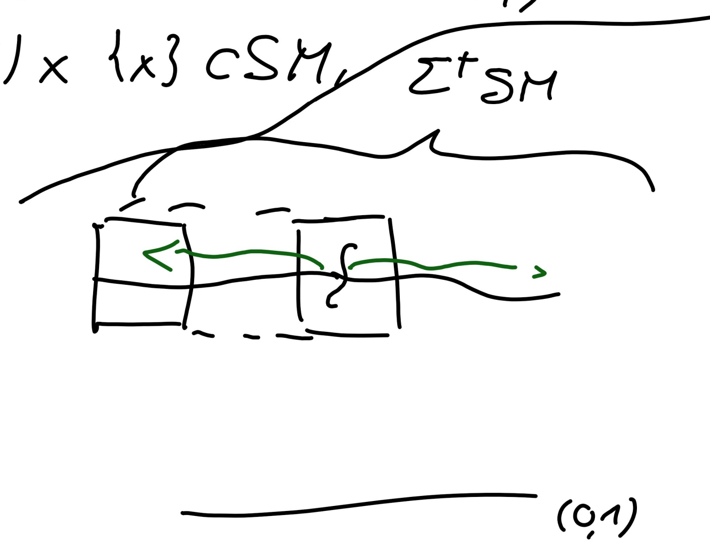
There are bundle isometries

$$\Sigma^\pm SM\big|_{M \times \{x\}} \cong \Sigma M$$

Let $U = (0,1) \times M$

Using parallel transport of sections in $C^\infty(\Sigma M)$ along geodesic lines $(0,1) \times \{x\} \subset SM$,

we get isometries



$$L^2(U, \Sigma^\pm SM, dvol_{Sg}) \xrightarrow[\sim]{\phi_\pm}$$

$$\xrightarrow{\sim} L^2((0,1), L^2(M, \Sigma M, \sin^n(r)\, dvol_g))$$

s.t.

$$\phi_- \circ D^+ \circ \phi_-^{-1} = \partial_r + \frac{n}{2}\cot(r) + \frac{1}{\sin(r)} D_M$$

compose $\phi_\pm$ with mult. with $\sin^{n/2}(r)$

$$(u,v)_{L^2} = \int_0^1 \int_M \langle u, v\rangle \sin^n(r)\, dvol_g\, dr$$

$$\rightsquigarrow L^2(U, \Sigma^\pm SM, dvol_{Sg})$$

$$\xrightarrow[\sim]{\tilde{\phi}_\pm} L^2((0,1), \underbrace{L^2(M, \Sigma M; dvol_g)}_{=: L})$$

and

Dirac op. on M

$$\tilde{\phi}_- \circ D^+ \circ \tilde{\phi}_+^{-1} = \partial_r + \frac{1}{\sin(r)} D_M$$

$$= \partial_r + \frac{1}{r} D_M + \underbrace{\left(\frac{1}{\sin(r)} - \frac{1}{r}\right) D_M}_{\text{pert. term}}$$

$\rightsquigarrow$ construct parametrix for D^+

$P : L^2(\Sigma^+ SM) \longrightarrow dom(\overline{D^+})$ bounded s.t.

$P\overline{D}^+ = id + L \qquad \overline{D}^+ P = id + R$

compact operators

existence of parametrix for $\overline{D^+}$

$\Leftrightarrow$ $\overline{D}^+$ Fredholm

deform. argument

$\Rightarrow$ $\exists$ harm. spinor

+ index formula

of Chou '85

Furthermore, integrated SL-formula still holds for mfds. with cone-like singularities