Talk1: • $\chi: \mathrm{End}_0 R \longrightarrow W(R)$ R commutative

• $W(R)$ for R non-commutative

& $\chi: \mathrm{End}_0(R) \longrightarrow W(R)$ (DKNP)

Talk2: $\chi: \mathrm{End}_0(R) \longrightarrow W(R)$ as a completion

• §1 <u>characteristic map (commutative rings)</u>

Let A be a commutative ring

R be a non-commutative ring.

• any matrix $M \in M_n(A)$, we can define characteristic polynomial.

$$\chi(B) = \det(I_n - tM)$$

$$\updownarrow t^n$$

$$\det(tI_n - M)$$

One can generalize this. (P,f) where P is f.g proj A-mod ($P \in \underline{P}(A)$) and $f: P \longrightarrow P$ is A-module homomorphism.

[$\underline{\underline{\mathrm{End}}}(A)$ obj: (P,f) $P \in \underline{P}(A)$, $f: P \to P$

mor: $(P,f) \xrightarrow{h} (Q,g)$

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[d, "g"] \\ P \arrow[r, "h"'] & Q \end{tikzcd}$$
]

* Take any Q st $P \oplus Q \cong A^m$

$f \oplus 0_Q$

$$\chi_f = \chi(P,f) := \chi(\underset{A^m}{\underset{\wr\|}{P\oplus Q}}, \underset{\text{matrix}}{\underset{\downarrow}{f \oplus 0_Q}}).$$

(need to show this independent of choice of Q)

• $\chi: (\mathrm{ob}\,\underline{\underline{\mathrm{End}}}(A), \oplus) \longrightarrow (1 + tA[[t]], \cdot)$

set map, monoid map

$$\chi(f \oplus g) = \chi(f)\chi(g)$$

$$\chi(hfh^{-1}) = \chi(f)$$

$$\chi(f \circ g) = \chi(g \circ f).$$

group completion. group of units of power series $A[[t]]$

$$\chi: K_0(\underline{\underline{\mathrm{End}}}(A)) \longrightarrow (1 + tA[[t]], \cdot)$$

$$[P,f] - [Q,g] \longmapsto \chi(f)\chi(g)^{-1}$$

abelian group homomorphism.

• $\otimes_A: \underline{\underline{\mathrm{End}}}(A) \times \underline{\underline{\mathrm{End}}}(A) \longrightarrow \underline{\underline{\mathrm{End}}}(A)$

$(P,f), (Q,g) \longmapsto (P \otimes_A Q, f \otimes_A g)$

$\otimes_A$ is associative, symmetric upto isom.

$\otimes: K_0(\underline{\underline{\mathrm{End}}}(A)) \times K_0(\mathrm{End}_0(A)) \longrightarrow K_0(\underline{\underline{\mathrm{End}}}(A))$

associative, commutative.

• $K_0(\underline{\underline{\mathrm{End}}}(A))$ is a commutative ring. multiplicative identity is $[A, 1]$.

$[P,0] \longleftarrow P$

• $\underline{\underline{\mathrm{End}}}(A) \rightleftarrows \underline{P}(A)$

$(P,f) \longmapsto P$

$$K_0(\underline{\underline{\mathrm{End}}}(A)) = \underset{\underset{K(A)}{\|}}{K_0(\underline{P}(A))} \oplus \mathrm{End}_0(A).$$

ring. ring.

$$\mathrm{End}_0(A) := \mathrm{Ker}(K_0(\underline{\underline{\mathrm{End}}}(A)) \xrightarrow{} K_0(A))$$

abelian group hom.

witt vector

$\chi: \mathrm{End}_0(A) \longrightarrow 1+tA[[t]] =: W(A)$

ring. also a ring. and χ is ring hom.

Almkvist showed:

$[f]-[g] \longmapsto \chi(f)\chi(g)^{-1}$

$= \dfrac{1+a_0t+\cdots+a_nt^n}{1+b_0t+\cdots+b_mt^m}$

image of χ all rational polynomials. $W^{rat}(A)$

- χ is injective.

$\chi: \mathrm{End}_0(A) \xrightarrow{\cong} W^{rat}(A) \hookrightarrow W(A)$

- $\underline{\underline{\mathrm{Nil}}}(A) :=$ objs (P,ν) $P \in \underline{P}(A)$

$\nu: P \to P$

ν nilpotent i.e $\nu^n = 0$ for some n.

$\cap\!|$

$\underline{\underline{\mathrm{End}}}(A)$

- $\underline{\mathrm{End}}(A) \times \underline{\mathrm{Nil}}(A) \longrightarrow \underline{\mathrm{Nil}}(A)$

$(P,f) \quad (Q,g) \longmapsto (P \otimes_A Q, f \otimes_A g)$

$g^n = 0 \Rightarrow (f\otimes g)^n = f^n \otimes g^n = 0$.

$\mathrm{Nil}_0(A) = \mathrm{Ker}(K_0(\underline{\underline{\mathrm{Nil}}}(A)) \to K_0(A))$

$\mathrm{End}_0 A \times \mathrm{Nil}_0 A \to \mathrm{Nil}_0 A$.

$\Rightarrow \mathrm{Nil}_0(A) \in \mathrm{End}_0 A$-module.

§2 Witt vectors (commutative)

$W(A) = 1+tA[[t]]$.

addition : $\cdot$

additive identity : 1

multiplication : $*$

multiplicative identity : $1-t$

$*$ is the unique continuous functorial operation st

$(1-at)*(1-bt) = 1-abt$

$(1-at^m) * (1-bt^n)$

$= (1-a^{n/d} b^{m/d} t^{mn/d})^d \quad d=(m,n)$

homothety.

- $[r] : 1-at^n \longmapsto 1-art^n$. $r \in A$.

$V_m : 1-at^n \longmapsto 1-at^{nm}$, $m \geq 1$

(Verschiebung)

$F_m \quad : 1-at^n \longmapsto (1-a^{m/d}t^{n/d})^d$.

(Frobenius) $d=(m,n)$.

- any element $\alpha(t) \in W(A)$.

$\alpha(t) = (1-a_1t)(1-a_2t^2)\cdots$

$= \prod_{i=1}^{\infty}(1-a_it^i)$

a_i's are unique.

- $\alpha(t) * \_ = (\sum V_m [a_m] F_m)(\_)$

§3: χ & W (commutative).

Grayson: $\overset{F_m, V_m, \otimes}{\circlearrowright}$, $\overset{F_m, V_m, *}{\circlearrowright}$

$$\chi : \mathrm{End}_0(A) \to W(A)$$

$$\underline{\mathrm{End}}(A) \to \underline{\mathrm{End}}(A)$$

$$(P,f) \xmapsto{[r]} (P, rf)$$

$$(P,f) \xmapsto{F_m} (P, f^m)$$

$$(P,f) \xmapsto{V_m} \left(P^{\oplus m}, \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & f \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}\right)$$

$$F_m, V_m : \mathrm{End}_0(A) \to \mathrm{End}_0(A).$$

Thm (Grayson):
χ commutes with F_m & V_m and respects the product.

$$(1-at^2) * (1-bt^2)$$

$$= \chi\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} * \chi\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

$$= \chi\left(\begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix}\right)$$

$$= \chi\begin{pmatrix} 0 & 0 & 0 & ab \\ 0 & 0 & b & 0 \\ 0 & a & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$= (1-abt)^2$$

- $\chi : \mathrm{End}_0 A \to W(A).$
$$= 1 + tA[[t]].$$

give $W(A)$ t-adic topology.

$$W^{(n)}(A) = 1 + t^n A[[t]].$$

$$W^{(1)}A \supset W^{(2)}(A) \supset \cdots$$

$$\varprojlim W(A)/W^{(n)}(A) \cong W(A)$$

- Give $\mathrm{End}_0(A)$ a t-adic topology.

$$I^{(n)}(A) := \chi^{-1}(W^{(n)}(A))$$

$$\varprojlim \mathrm{End}_0(A)/I^{(n)}(A) = W(A)$$

* Witt vector is the completion of $\mathrm{End}_0(R)$ via χ.

(Steinstra) Then hold for $\mathrm{End}_0(R)$.

- $F_m V_m = m \cdot 1$
- $F_m F_n = F_{mn}$, $V_m V_n = V_{mn}$
- $F_m(\alpha * \beta) = F_m\alpha * F_m\beta$.
- $V_m(\alpha * F_m\beta) = V_m\alpha * \beta$.

(Steinstra):

- $\mathrm{End}_0 A \times \mathrm{Nil}_0 A \to \mathrm{Nil}_0 A$.

$\rightsquigarrow$

- $W(A) \times \mathrm{Nil}_0 A \to \mathrm{Nil}_0 A$.

$\mathrm{Nil}(A) \in W(R)$-mod.

- $1 + a_1 t + a_2 t^2 + \cdots + a_n t^n$

$\rightsquigarrow$

$$\begin{pmatrix} 0 & 0 & \cdots & & -a_n \\ 1 & 0 & & & -a_{n-1} \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & & & \\ 0 & & & 1 & -a_1 \end{pmatrix} \circlearrowleft x$$

$1 - at^n$.

$$\begin{pmatrix} 0 & \cdots & & a \\ 1 & \ddots & & 0 \\ & & \ddots & \vdots \\ 0 & & 1 & 0 \end{pmatrix} = V_m[A, a]$$

(§3) Witt vector (non-commutative) DKNP.

$W(R, M)$ $\quad M \in R$-R bimodule

$$T(R,M) = \prod_{n \geq 0} M^{\otimes_R n}.$$

$a_0 + a_1 t + a_2 t^{(2)} + \cdots$

$a_i \in \underbrace{M \otimes_R M \otimes_R \cdots \otimes_R M}_{i \text{ time}}$.

give $T(R,M)$ the product topology. then continuous ring etc with multiplication.

$(a_0 + a_1 t + a_2 t^2 + \cdots)(b_0 + b_1 t + \cdots)$

$= [a_0 \otimes b_0 + (a_0 \otimes b_1 + a_1 \otimes b_0) t + \cdots$

$T(R,M) \rightsquigarrow A[[t]]$

$S(R,M) \rightsquigarrow 1 + tA[[t]]$.

$S(R,M)$ is the special unit; multiplicative subgroup of const term is 1.

Def: $W(R,M) = \dfrac{S(R,M)}{\overline{N}}$

$\| \quad S(R,M)^{ab} / \tau(rm) \sim \tau(mr), \; r \in R \quad m \in M^{\otimes n}$

N is the normal subgroup generated by $(1-tfg)(1-tgf)^{-1}$, $f,g \in T(R,M)$

<u>Trace property</u> :-

$R_1 \quad R_2 \quad R_3$ rings

$M_{1,2} \quad M_{2,3} \quad M_{3,1}$

$$W(R_1, \underbrace{M_{1,2} \otimes_{R_2} M_{2,3} \otimes_{R_3} M_{3,1}}_{R_1\text{-}R_1 \text{ bimodule.}})$$

$$\cong$$

$$W(R_2, \underbrace{M_{2,3} \otimes_{R_3} M_{3,1} \otimes_{R_1} M_{1,2}}_{R_2\text{-}R_2 \text{ bimod}})$$

(DKNP) $F : \underline{P}(R) \to \underline{P}(S)$ additive

$\Big\}$ induces

$$W(F) : W(R) \to W(S)$$

<u>cor</u> : R & S are Morita equivalent

$$\underline{P}(R) \underset{G}{\overset{F}{\rightleftarrows}} \underline{P}(S)$$

$$W(R) \underset{W(G)}{\overset{W(F)}{\underset{\cong}{\rightleftarrows}}} W(S).$$

$W(-)$ is Morita invariant

$\tau(m) = 1 - tm$.

- (DKNP)

$$\chi : End_0(R) \longrightarrow \underline{W(R)}.$$

need trace property.

- $End_i(R) := \ker \left(K_i(\underline{End}(R)) \to K_i(R) \right)$

$End_i(R) \times End_j R \to End_{i+j}(R)$

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