Part II

I Characteristic map (DKNP)

- $W(R, \underset{R-S}{M_1} \otimes_S \underset{S-R}{M_2})$

$\cong W(S, \underset{S-R}{M_2} \otimes_R \underset{R-S}{M_1})$

- $W(-)$ is Morita invariant.
$F : \underline{P}(R) \to \underline{P}(S)$ additive

$\Downarrow$ induces

$W(F) : W(R) \to W(S)$.

- characteristic element of an $[P,f]$ $f : P \to P$.
χ_f

$End_R(P)$ all endomorphisms $P \to P$.

$- \otimes_{End_R(P)} P : \underline{P}(End_R(P)) \to \underline{P}(R)$

$\Downarrow$

$$W(End_R(P)) \xrightarrow{\beta'} W(R)$$

$\chi_f : End_R(P) \to W(End_R P) \xrightarrow{\beta} W(R)$
$(P,f) \mapsto 1 - tf \mapsto \chi_f$

- $\chi_{f \oplus g} = \chi_f \cdot \chi_g$ in $W(R)$
$\chi_{f \circ g} = \chi_{g \circ f}$
$\chi_{h^{-1} f h} = \chi_f$

- $\chi : End_0(R) \to W(R)$.

- witt vectors have $F_m, V_m, \underline{*}$
$V_n V_m = V_{nm}$
$F_n F_m = F_{nm}$.
$F_n V_m = V_m F_n$.
$F_n(x * y) = F_n(x) * F_n(y)$
$\underline{\underline{V_n(x * F_n(y))}} = V_n(x) * y$

$* : W(R) \times W(S) \to W(R \otimes_{\mathbb{Z}} S)$

if $R = A$ comm.
$A \otimes_{\mathbb{Z}} A \xrightarrow{\mu} A$ is a ringhom.
$W(\quad) \longrightarrow W(A)$

II. Alternate def^n of χ

Almkvist
$\chi : End_0(A) \hookrightarrow W(A)$
$img(\chi) = \left\{ \frac{1 + a_1 t + \cdots + a_n t^n}{1 + b_1 t + \cdots + b_m t^m} \right\}$

If you give $W(A) = 1 + tA[[t]]$ the t-adic top then $img(\chi)$ is dense in $W(A)$

$W(R) := W(R,R)$

$W(A) \supset W^{(1)}(A) \supset \cdots \supset W^{(n)}(A) \supset \cdots$

$1 + t^n A[\![t]\!]$

complete $W(A)$ with this filtration. you get back $W(A)$.

- $\mathrm{img}(\chi) \supset \mathrm{img}(\chi) \cap W^{(1)}(A) \supset \cdots$

complete $\mathrm{img}(\chi)$, you get back $W(A)$

$$\mathrm{img}(\chi)^\wedge = W(A).$$

$$I^{(n)}(A) = \chi^{-1}(W^{(n)}(A))$$

$$\mathrm{End}_0(A) \supset I^{(0)}(A) \supset I^{(1)}(A) \supset \cdots$$

$$\mathrm{End}_0(A)^\wedge = W(A).$$

- $A \supset F^{(1)}A \supset F^{(2)}A \supset \cdots$

$$A^\wedge = \lim A / F^{(n)}A.$$

- (Shaham)

$$\theta^R : \mathrm{End}_0(R) \to \tilde{K}_1(R[\![t]\!])$$
$$[P, f] \mapsto [1 - ft]$$

$1 + ft + f^2t^2 + \cdots$

$*$ $K_1(R) = GL(R)^{ab}$

$1 - ft : R[\![t]\!] \to R[\![t]\!] \in GL_1(R[\![t]\!])$

- $1 + tR[\![t]\!] \xrightarrow{i} \tilde{K}_1(R[\![t]\!])$.

and kernel is N gen by $(1-tpq)(1-tqp)^{-1}$ $p, q \in R[\![t]\!]$.

$$\frac{1 + tR[\![t]\!]}{N} \cong \tilde{K}_1(R[\![t]\!]).$$

$$\frac{1 + tR[\![t]\!]}{\overline{N}} = W(R,R) =: W(R)$$

$\chi : \mathrm{End}_0(R) \xrightarrow{\theta^R} \tilde{K}_1(R[\![t]\!]) \xrightarrow{\det} W(R)$

is alternate defn of characteristic map.

$$\begin{array}{ccc}
\dfrac{1+tR[\![t]\!]}{N} & \xrightarrow{q} & \dfrac{1+tR[\![t]\!]}{\overline{N}} \\
\wr\wr & & \wr\wr \\
\tilde{K}_1(R[\![t]\!]) & \overset{\det}{\dashrightarrow} & W(R) \\
\| & & \uparrow \\
GL(R[\![t]\!])^{ab} & \to & \dfrac{1+tR[\![t]\!]}{\overline{N}}
\end{array}$$

- $W(R) \supset \cdots \supset W^{(n)}(R) \supset \cdots$

$$\begin{array}{ccc}
 & & \| \\
1 + t^n R[\![t]\!] & \dashrightarrow & W^{(n)}(R) \\
\cap & & \cap \\
1 + tR[\![t]\!] & \xrightarrow{\text{quotient}} & W(R)
\end{array}$$

* $W(R)^\wedge = W(R)$

* $\theta_n^R : \mathrm{End}_0(R) \longrightarrow \tilde{K}_1(R[t]/t^n)$
  $[P,f] \longmapsto [1-tf]$.

$I^{(n)}(R) := \chi^{-1}(W^{(n)}(R)) = \underline{\underline{\mathrm{Ker}\,\theta_n^R}}$.

$$\begin{tikzcd}
\mathrm{End}_0(R) \arrow[r, "\theta^R"] \arrow[d, "\theta_n^R"'] \arrow[dr, dashed, "\chi"] & \tilde{K}_1(R[[t]]) \arrow[d, "\det"] \\
 & W(R) \arrow[d, "a_n"] \\
\tilde{K}_1(R[t]/t^n) \arrow[r, "\cong"] & W(R)/W^{(n)}R
\end{tikzcd}$$

$\longrightarrow \chi^{-1}(W^{(n)}R)$
$= \mathrm{Ker}(\underline{\underline{a_n \circ \chi}})^{-1}$
$= \mathrm{Ker}(\theta_n^R)$

* $\mathrm{End}_0(R)^\wedge \cong \underline{W(R)}$
* $\mathrm{Ker}(\chi) = \bigcap_n I_R^{(n)}$.

III Recovering everything as a K-thy phenomenon

$\underline{F} : \underline{P}(R) \to \underline{P}(S)$ additive.

$$\begin{array}{c} \rightsquigarrow \\ \underline{\mathrm{End}}(R) \longrightarrow \underline{\underline{\mathrm{End}}}(S) \\ (P,f) \longmapsto (F(P), F(f)) \end{array}$$

- $\phi : \mathrm{End}_0(R) \to \mathrm{End}_0(S)$

$I_R^{(n)}$ $\quad I_S^{(n)}$ ✓

$* \ \phi(I_R^{(n)}) \subseteq I_S^{(n)}$ ✓✓

$\phi^\wedge : (\mathrm{End}_0 R)^\wedge \longrightarrow \mathrm{End}_0(S)^\wedge$

$W(\phi) \quad W(R) \dashrightarrow W(S)$

$W(\phi) \circ \chi^R = \chi^S \circ \phi$

Cor: $R \underset{\text{morita}}{\simeq} S$ then $W(R) \cong W(S)$

$G : \underline{P}(R) \times \underline{P}(S) \longrightarrow \underline{P}(T)$

- $F_m, V_m : \underline{\underline{\mathrm{End}}}(R) \to \underline{\underline{\mathrm{End}}}(R)$

$F_m(P,f) = (P, f^m)$

$V_m(P,f) = \left(P^{\oplus m}, \begin{pmatrix} 0 & \cdots & & f \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}\right)$

$F_m \circ \chi = \chi \circ F_m$

- (a)

$\chi(a) = 1 - at$

$F_m \chi(a) = F_m(1-at)$.

$\Rightarrow \chi \circ F_m(a) = F_m(1-at)$

$\Rightarrow \chi(a^m) = F_m(1-at)$

$\Rightarrow 1 - a^m t = F_m(1-at)$

Rmk: One can directly check $\chi \circ F_m = F_m \circ \chi$ using DRNP def^n

- On $\mathrm{End}_0(R)$

$V_m(x * F_m(y)) = V_m(x) * y$.

- (Ranicki, Sheiham) Cohn localization ring.

$\mathrm{End}_0(R) \xrightarrow{\cong} \tilde{K}_1(\Sigma^{-1} R[\![t]\!])$.

$[P,f] \longmapsto [1-tf]$

Σ is the set of all matrices in $R[\![t]\!]$ st if you put $\left(\begin{array}{c} R[\![t]\!] \xrightarrow{\varepsilon} R \\ t \longmapsto 0 \end{array}\right)$ $t=0$ then it becomes $\in GL(R)$.

- $\mathrm{End}_0(A) \xrightarrow{\cong} W^{rat}(A) \hookrightarrow W(A)$

$$\begin{array}{ccccc} \underline{\mathrm{End}_0(R)} & \xrightarrow{\cong} & W^{rat}(R) & \dashrightarrow & W(A) \\ & & \wr\| & & \wr\| \\ & & \tilde{K}_1(\Sigma_i^{-1} R[t]) & & \underline{\tilde{K}_1(R[t])} \end{array}$$

- (Row reduction)

- $\mathrm{End}_0 R \times \underline{\mathrm{Nil}_i R} \to \mathrm{Nil}_i(R \otimes_{\mathbb{Z}} R)$

$$\downarrow$$

$W(R) \times \mathrm{Nil}_0(R) \to \mathrm{Nil}_0(R \otimes_{\mathbb{Z}} R)$.

$\mathrm{Nil}_0 R$ $\quad$ $\mathrm{Fil}_{\underline{n}} \mathrm{Nil}_0 R =$ gen by $[P, \nu]$ st $\underline{\nu^n = 0}$

$$\underbrace{\mathrm{End}_0 R / I^{(n)} R}_{\downarrow} \times \underline{\mathrm{Fil}_n \mathrm{Nil}_0 R} \to \mathrm{Nil}_0(R \otimes R)$$

$\varprojlim$ $W(R)$ $\quad$ $\varinjlim$ $\mathrm{Nil}_0 R$

- $\mathrm{End}_i R = \ker(K_i(\underline{\mathrm{End}}(R)) \to K_i(R))$

$[P, f] \mapsto [P, 0]$.

$\mathrm{Nil}_i(R) \longrightarrow K_{i+1}(R[t])$