What is a Singular Foliation?

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Can you find the snake hiding in these dried leaves?

Figure 1: Copyright, Jerry Devis, X(Twitter)

Take a look closer!

Introduction

We are interested in: A partition of a manifold by submanifolds of possibly different dimensions (called singular foliations)

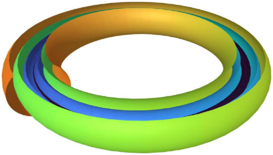

Figure 2: Self-eating-snake singular foliation on $S^1 \times \mathbb{R}^2$.

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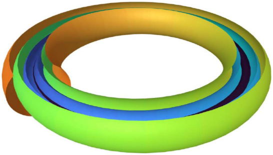

Figure 2: Self-eating-snake singular foliation on $S^1 \times \mathbb{R}^2$.

(Singular) foliations arose from the qualitative study of (partial) differential equations!

Introduction

In differential geometry, Singular foliations arise as

- Lie group actions
- Symplectic leaves of a Poisson manifold
- Orbits of a Lie groupoids or Lie algebroids.
- The list could continue...
- ...

Singular Foliations

Definition (Most used definition)

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1. **Stability under Lie bracket**: $[\mathscr{F}, \mathscr{F}] \subseteq \mathscr{F}$.
2. **Local finite generateness**: every $m \in M$ admits an open neighborhood $\mathcal{U} \subseteq M$ together with a finite number of vector fields $X_1, \ldots, X_r \in \mathfrak{X}(\mathcal{U})$ such that for every open subset $\mathcal{V} \subseteq \mathcal{U}$ the vector fields $X_{1|_\mathcal{V}}, \ldots, X_{r|_\mathcal{V}}$ generate $\mathscr{F}$ on $\mathcal{V}$ as a $\mathcal{O}_\mathcal{V}$-module.

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This definition includes the usual definition of (regular) foliation for $\mathscr{F} = \Gamma(F)$, the section of an involutive vector sub-bundle $F \subseteq TM$.

Singular foliations admit leaves

According to **Hermann's theorem, a singular foliation induces a partition of the manifold $M = \cup_\alpha L_\alpha$ into immersed submanifolds $L_\alpha \subseteq M$ of <u>possibly "different" dimensions</u>**, called <u>leaves</u> of the singular foliation.

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The leaf $L_m \subset M$ through $m \in M$ is:

$$\left\{ \phi_{t_1}^{X_1} \circ \phi_{t_2}^{X_2} \circ \cdots \circ \phi_{t_n}^{X_n}(m),\ t_1, \ldots, t_n \in \mathbb{R},\ X_1, \ldots, X_n \in \mathscr{F},\ n \in \mathbb{N} \right\}$$

Concentric spheres

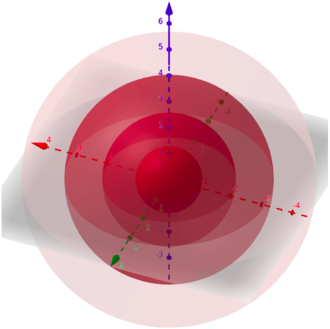

Figure 3: Concentric spheres foliation on $\mathbb{R}^3$.

Concentric spheres

Consider the Lie–Poisson structure on $M = \mathfrak{so}(3,\mathbb{R})^*$ given on a dual basis x, y, z by

$$\pi = -z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} - x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$

The singular foliation $\mathscr{F}_\pi$ is generated by the vector fields

$$X := \pi^\#(dx) = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad Y := \pi^\#(dy) = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},$$

and

$$Z := \pi^\#(dz) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Regular and singular points

Thank you for your attention!
See you next week!

References

Iakovos Androulidakis and Marco Zambon.
Stefan–sussmann singular foliations, singular subalgebroids and their associated sheaves.
International Journal of Geometric Methods in Modern Physics, 13(Supp. 1):1641001, 2016.

Camille Laurent-Gengoux, Ruben Louis, and Leonid Ryvkin.
An invitation to singular foliations.
In: Garcia-Fernandez, M., Ponte, D.I., Miranda, E., Oms, C., Rubio, R. (eds) Advances in Poisson Geometry. Advanced Courses in Mathematics - CRM Barcelona. Birkhäuser, Cham. https://doi.org/10.1007/978-3-031-86657-9$_3$, 2024.

Ruben Louis.
On symmetries of singular foliations.
Journal of Geometry and Physics, 189:104833, 2023.