Day 2

Recall: A singular foliation $(M, \mathcal{F})$, $\mathcal{F} \subset \mathfrak{X}(M)$

- $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$
- locally finitely generated ...

Ex: . $X \in \mathfrak{X}(M)$, $\mathcal{F} = \underset{C^\infty(M)}{\mathrm{Span}}(X)$ is a singular foliation

. G Lie group acting on M and $\mathfrak{g} = \mathrm{Lie}(G)$ and $\{v_1, \dots, v_n\}$ a basis of $\mathfrak{g}$ then every v_i defines a vector field $\overline{v}_i \in \mathfrak{X}(M)$

Take $\mathcal{F} = \underset{C^\infty(M)}{\mathrm{Span}}(\overline{v}_i,\ i = 1, \dots, n) \subset \mathfrak{X}(M)$

. Lie algebroid: $A \to M$ and Lie bracket

$$[\cdot,\cdot]_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$$

+ vector bundle morphism $\rho : A \to TM$ (anchor)

with $\rho([a,b]_A) = [\rho(a), \rho(a)]$

Leibniz: $[a, fb]_A = \rho(a)[f]\, b + f\,[a,b]_A$

$\rho(a) \in \mathfrak{X}(M)$

Every Lie algebroid gives rise to a singular foliation:

$$\mathcal{F} = \rho(\Gamma(A)) \subset \mathfrak{X}(M)$$

is singular foliation.

- Poisson manifold: $(M, \{\cdot,\cdot\})$ with

  $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ a Lie bracket

  such that $\forall f \in C^\infty(M)$, $X_f = \{f, \cdot\} \in \mathfrak{X}(M)$

  X_f: Hamiltonian vector field

  $$\mathcal{F} = \operatorname{Span}_{C^\infty(M)}\left(X_f,\ f \in C^\infty(M)\right) \subset \mathfrak{X}(M).$$

  is a singular foliation. Moreover, the leaves are symplectic manifolds. Called "the symplectic foliation of $(M, \{\cdot,\cdot\})$".

  → (N, ω), $\omega \in \Omega^2(N)$

  closed, $d\omega = 0$

  non-degenerated.

Singular & regular points

Let $(M, \mathcal{F})$ be a singular foliation. For $x \in M$,

$$I_x = \{ f \in C^\infty(M),\ f(x) = 0 \}$$

$$\underbrace{\{X \in \mathcal{F} \mid X(x) = 0\}}_{I_x \mathcal{F}} \hookrightarrow \underbrace{\mathcal{F}}_{I_x \mathcal{F}} \xrightarrow{ev_x} \underbrace{T_x \mathcal{F} = \{X(x),\ X \in \mathcal{F}\}}_{T_x L}$$

$$\mathfrak{g}_x \hookrightarrow \mathcal{F}_x \xrightarrow{ev_x} T_x L$$

where $L \subset M$ is a leaf at x.

Prop. $\forall x \in M$, $\mathfrak{g}_x$ is a Lie algebra called isotropy Lie algebra of $\mathcal{F}$ at x.

Def: The regular points of $(M, \mathcal{F})$ are those where $\mathfrak{g}_x = \{0\}$.

Lie algebroid & Poisson structures

Ted Courant:

Prop: Lie algebroid structure on $A \to M$ $\overset{1:1}{\longleftrightarrow}$ linear Poisson structure on A^*

$\{C^\infty_{lin}(A^*), C^\infty_{lin}(A^*)\} \subset C^\infty_{lin}(A^*)$

$C^\infty_{lin}(A^*) = \{ F: A^* \to \mathbb{R}, \text{ smooth} \mid F|_{A^*_x} \text{ is linear} \}$

This correspondence is given as follows:

$$\Gamma(A) \xrightarrow{\sim} C^\infty_{lin}(A^*) \hookrightarrow C^\infty(A^*)$$
$$a \longmapsto ev_a : A^* \to \mathbb{R}$$
$$\xi \longmapsto \langle \xi, a \circ \pi(\xi) \rangle$$

where $\pi : A^* \to M$.

$$\boxed{\{ev_a, ev_b\} = ev_{[a,b]_A}, \quad \forall a, b \in \Gamma(A)}$$

In particular: for $a \in \Gamma(A)$, $f, g \in C^\infty(M)$

$$\{ev_a, \pi^* f\} = \pi^*(\rho(a)[f])$$

$$\{\pi^* f, \pi^* g\} = 0.$$

Cor: $M = \{*\}$, $A = \mathfrak{g}$ is a Lie algebra

$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}) \xleftrightarrow{1:1}$ linear Poisson structure on $\mathfrak{g}^*$

Cor: Regular foliations are Lie algebroids with injective anchor map:

$F \subset TM$ (involutive vector subbundle)

What about singular foliations?

$\mathcal{F} \subset \mathfrak{X}(M)$

↯ is not the module of sections of any vector bundle in general!

Lie-Rinehart algebras

Every Lie algebroid gives rise to a Lie-Rinehart algebra

Lie algebroid

$E \to M$, $[\cdot,\cdot]_E$, $\rho: E \to TM$

Lie-Rinehart algebra

$(\Gamma(E), [\cdot,\cdot]_E)$ ← Lie algebra

$\rho: \Gamma(E) \to \mathfrak{X}(M) \simeq \mathrm{Der}(C^\infty M)$

+ Leibniz rule!

Projective module! (locally free!)

Q: what if the r.h.s is not projective?

(this occur very often in) DG

Ex: Singular foliations

Observation: (our approach to geometrizing arbitrary modules)

1. For a v.b $H \xrightarrow{p} M$, the map

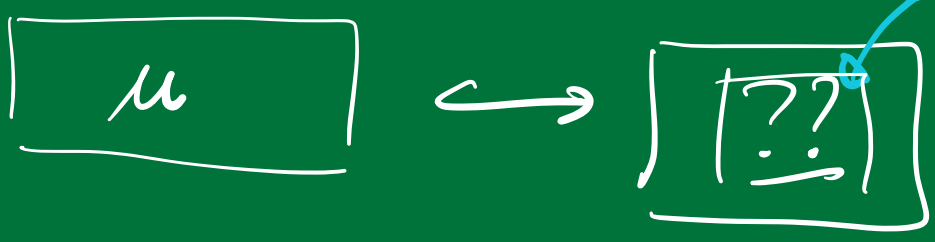
$$p^*: C^\infty(M) \longrightarrow C^\infty(H)$$

gives $C^\infty(H)$ a $C^\infty(M)$-algebra structure.

2. For a v.b $E \to M$ and $E^* \xrightarrow{\pi} M$ its dual

$$\Gamma(E) \longrightarrow \boxed{C^\infty(E^*)} \qquad C^\infty(M)\text{-algebra!}$$
$$A \longmapsto ev_s$$

Q: For $C^\infty(M)$-module $\mathcal{M}$



$$C^\infty(M)\text{-Mod} \xrightarrow{F} C^\infty(M)\text{-Alg}$$

Goal: For an arbitrary $C^\infty(M)$-module $\mathcal{M}$ construct a "smooth" analog of $C^\infty(E^*)$

C^∞-rings

Remark: $C^\infty(M)$ this not just an $\mathbb{R}$-algebra!

Ex: $a \in C^\infty(M) \Rightarrow e^a = \exp \circ a \in C^\infty(M)$

$e^x \in C^\infty(\mathbb{R})$

↑

Cannot be expressed algebraically in terms of the $\mathbb{R}$-algebra structure!

We need more: $C^\infty(M)$ is a C^∞-ring!

Lie-Rinehart and Poisson algebras over C^∞-rings

Def. A C^∞-ring is a product preserving functor

Objects: $\mathbb{R}^n$, $n \geq 0$
Arrows: $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$

$\longrightarrow$ Set

Equivalently, a C^∞-ring is a set A together with a collection of n-ary operations, $n \geq 0$

$\forall\, f \in C^\infty(\mathbb{R}^n)$, $\quad f_A : A^n \longrightarrow A$.

+

- Associativity
- If $x_i : \mathbb{R}^n \to \mathbb{R}$, is the ith projection then

$(x_i)_A : A^n \to A$ is given by

$$(x_i)_A(a_1, \dots, a_n) = a_i.$$

Ex: $A = C^\infty(M) = C^\infty(M, \mathbb{R})$ is a C^∞-ring

$\forall\, n \geq 0$, and $f \in C^\infty(\mathbb{R}^n)$ define

$$f_A : C^\infty(M)^n \longrightarrow C^\infty(M)$$
$$(a_1, \dots, a_n) \longmapsto f \circ (a_1, \dots, a_n)$$

Remark: A C^∞-ring is in particular an $\mathbb{R}$-algebra:

- Addition: $f_A : A^2 \to A$, with $f : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x+y$
- Multiplication: $f(x,y) = xy$
- Scalar multiplication: $\lambda \in \mathbb{R}$, $f_A : A \to A$ with $f(x) = \lambda x$.

Ex: $\cdot\ \mathcal{A} = \dfrac{C^\infty(\mathbb{R})}{\langle x^2 \rangle} \simeq \{a + \varepsilon b,\ \varepsilon^2 = 0,\ a, b \in \mathbb{R}\}$

is C^∞-ring

$\cdot\ \mathcal{A} = \dfrac{C^\infty(\mathbb{R}^2)}{\langle xy \rangle}$ is C^∞-ring

- $\forall n \in \mathbb{N}$, $\mathcal{A} = C^\infty(\mathbb{R}^n)$ is free C^∞-ring generated by x_i, $i = 1, \dots n$

$\forall f \in C^\infty(\mathbb{R}^n)$, $f = f(x_1, \dots, x_n) = \underset{\mathcal{A}}{f}(x_1, \dots, x_n)$.

- For any set I

$$C^\infty(\mathbb{R}^I) := \left\{ f : \mathbb{R}^Y \to \mathbb{R} \mid \exists\ X \subseteq Y \text{ finite and } \tilde{f} \in C^\infty(\mathbb{R}^X) \text{ such that } f = \tilde{f}|_{\mathbb{R}^X} \right\}$$

is free C^∞-ring