LIE-RINEHART AND POISSON ALGEBRAS OVER C^∞-RINGS

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Vector bundles and modules

On the blackboard!

Lie algebroids and Lie-Rinehart algebras

On the blackboard!

Geometrizing arbitrary modules

Question

How can one geometrize arbitrary modules, not necessarily projective?

Example: singular foliations and singular Lie subalgebroids !

Observation

On the blackboard!

Why C^∞-rings?

$C^\infty(M)$ is more than a commutative $\mathbb{R}$-algebra

For every smooth map $f : \mathbb{R}^n \to \mathbb{R}$, there is an operation

$$\Phi_f : C^\infty(M)^n \to C^\infty(M)$$

given by

$$\Phi_f(a_1, \ldots, a_n)(m) = f(a_1(m), \ldots, a_n(m)).$$

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Example

$$a \in C^\infty(M),\ e^x \in C^\infty(\mathbb{R}) \quad \Longrightarrow \quad e^a \in C^\infty(M).$$

This operation CAN NOT be recovered from the ordinary $\mathbb{R}$-algebra structure alone.

C^∞-rings

Definition

A C^∞-ring is a product-preserving functor

$$\mathbf{Euc} \longrightarrow \mathbf{Set},$$

The objects of $\mathbf{Euc}$ are $\mathbb{R}^n$, $n \in \mathbb{N}$ and the morphisms are smooth maps.

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Equivalently, a C^∞-ring is a set $\mathscr{A}$ equipped with operations

$$f_{\mathscr{A}} : \mathscr{A}^n \to \mathscr{A}, \qquad f \in C^\infty(\mathbb{R}^n), \quad n \geq 0,$$

satisfying the natural compatibility conditions:

- Associativity
- If $x_i : \mathbb{R}^n \to \mathbb{R}$ is the i-th projection so is

$$(x_i)_{\mathscr{A}} : \mathscr{A}^n \to \mathscr{A}.$$

C^∞-rings

Remark

A C^∞-ring is in particular an $\mathbb{R}$-algebra with

- **Addition**: $f(x, y) = x + y$
- **Multiplication**: $f(x, y) = xy$
- **Scalar multiplication**: $\lambda \in \mathbb{R}$, $f(x) = \lambda x$

Examples

On the blackboard!

The "**free**" C^∞-ring on n generators is $C^\infty(\mathbb{R}^n)$. It is generated by the coordinate functions $x_1, \ldots, x_n$.

C^∞-derivations

Let $\mathscr{A}$ be a C^∞-ring and $\mathscr{M}$ an $\mathscr{A}$-module.

Definition

A C^∞-derivation with values in $\mathscr{M}$ is a $\mathbb{R}$-linear map

$$X : \mathscr{A} \to \mathscr{M}$$

such that, for every $f \in C^\infty(\mathbb{R}^n)$ and $a_1, \ldots, a_n \in \mathscr{A}$,

$$X\big(f_{\mathscr{A}}(a_1, \ldots, a_n)\big) = \sum_{i=1}^{n} (\partial_i f)_{\mathscr{A}}(a_1, \ldots, a_n) \cdot X(a_i)$$

We denote the space of such derivations by $C^\infty\mathrm{Der}(\mathscr{A}, \mathscr{M})$.
If $\mathscr{M} = \mathscr{A}$, we write $C^\infty\mathrm{Der}(\mathscr{A})$.

C^∞-derivations

Important

One has

$$C^\infty \mathrm{Der}(\mathscr{A}, \mathscr{M}) \subsetneq \mathrm{Der}(\mathscr{A}, \mathscr{M}),$$

Issue

- Osborn 69': in $\Omega^1_{C^\infty(\mathbb{R}),alg}$

$$d^{alg}(\sin t) \neq \cos(t) d^{alg} t$$

- The case $\mathscr{A} = \mathscr{M}$: Lerman–Louis, arxiv 2026

Poisson C^∞-rings

Definition

A Poisson C^∞-ring is a C^∞-ring $\mathscr{A}$ together with a Poisson bracket

$$\{\cdot\,,\cdot\} : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$$

on the underlying commutative $\mathbb{R}$-algebra such that

$$\{a, \cdot\} \in C^\infty\mathrm{Der}(\mathscr{A})$$

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Example

If M is a Poisson manifold, then

- $C^\infty(M)$ is a Poisson C^∞-ring.
- If $\mathcal{I} \subset C^\infty(M)$ is a Poisson ideal, then $C^\infty(M)/\mathcal{I}$ is a Poisson C^∞-ring.

Lie-Rinehart algebras over C^∞-rings

Definition

A Lie–Rinehart algebra over a C^∞-ring $\mathscr{A}$ is an $\mathscr{A}$-module $\mathscr{M}$, equipped with

$$[\cdot\,,\cdot]_{\mathscr{M}} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$$

and an $\mathscr{A}$-linear map (anchor)

$$\rho : \mathscr{M} \to C^\infty\mathrm{Der}(\mathscr{A}),$$

such that:

1. $(\mathscr{M}, [\cdot\,,\cdot]_{\mathscr{M}})$ is an $\mathbb{R}$-Lie algebra;
2. ρ is a morphism of $\mathbb{R}$-Lie algebras;
3. the Leibniz rule holds:

$$[m_1, am_2]_{\mathscr{M}} = \rho(m_1)(a)m_2 + a[m_1, m_2]_{\mathscr{M}}.$$

Basic examples

Example

- If $\mathscr{A}$ is a C^∞-ring, then

$$\mathrm{id}\colon C^\infty\mathrm{Der}(\mathscr{A}) \to C^\infty\mathrm{Der}(\mathscr{A})$$

  with the commutator bracket and the identity anchor.
- If $E \to M$ is a Lie algebroid, then

$$\Gamma(E) \to C^\infty\mathrm{Der}(C^\infty(M)) = \mathfrak{X}(M)$$

  is a Lie–Rinehart algebra.
- Singular foliations provide Lie–Rinehart algebras whose underlying modules are generally not projective.

(Smooth) Kähler differentials

Let $\mathscr{A}$ be a C^∞-ring.

Dubuc and Kock : There exists an $\mathscr{A}$-module $\Omega^1_{\mathscr{A}}$ and a universal C^∞-derivation

$$d \equiv d_{\mathscr{A}} : \mathscr{A} \to \Omega^1_{\mathscr{A}}$$

such that, for every $\mathscr{A}$-module $\mathscr{M}$,

$$C^\infty\mathrm{Der}(\mathscr{A}, \mathscr{M}) \simeq \mathrm{Hom}_{\mathscr{A}}(\Omega^1_{\mathscr{A}}, \mathscr{M}).$$

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Universal property

Every C^∞-derivation $X : \mathscr{A} \to \mathscr{M}$ factors uniquely through $d_{\mathscr{A}}$:

$$\mathscr{A} \xrightarrow{d_{\mathscr{A}}} \Omega^1_{\mathscr{A}} \longrightarrow \mathscr{M}.$$

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For any manifold M the module of Kähler differentials $\Omega^1_{C^\infty(M)}$ is the module of de Rham 1-forms $\Omega^1(M)$.

First main theorem

Theorem (Lerman–Louis 26')

Let $(\mathscr{A}, \{\cdot\,,\cdot\})$ be a Poisson C^∞-ring. Then the $\mathscr{A}$-module of C^∞-Kähler differentials is naturally a Lie–Rinehart algebra

$$\rho : \Omega^1_{\mathscr{A}} \to C^\infty\mathrm{Der}(\mathscr{A}), \quad \rho(\sum a_i db_i) = \sum a_i \{b_i, \cdot\}$$

Tomorrow

Plan

1. Motivation: vector bundles, modules, and singular geometry
2. C^∞-rings and Poisson C^∞-rings
3. Lie–Rinehart algebras over C^∞-rings
4. Free "symmetric" algebras over C^∞-rings
5. Lie–Rinehart algebras and "linear" Poisson structures

To be continued...

Free "Symmetric algebra" over a C^∞-ring $\mathscr{A}$

Second main theorem

Definition

An $\mathscr{A}$-algebra is a C^∞-ring $\mathscr{B}$ together with a morphism

$$\varphi : \mathscr{A} \to \mathscr{B}$$

of C^∞-rings (Operations preserving).

There is a forgetful functor

$$U : \mathscr{A}/C^\infty\text{Ring} \to \mathscr{A}\text{-Mod}.$$

Theorem (Lerman–Louis 26')

The functor U admits a left adjoint (computable)

$$\mathcal{F} : \mathscr{A}\text{-Mod} \to \mathscr{A}/C^\infty\text{Ring}.$$

Universal property of $\mathcal{F}(\mathscr{M})$

For every $\mathscr{A}$-module $\mathscr{M}$, there is an $\mathscr{A}$-algebra $\mathcal{F}(\mathscr{M})$ and

- a map $\varphi_{\mathscr{M}} : \mathscr{A} \to \mathcal{F}(\mathscr{M})$ of C^∞-rings
- an $\mathscr{A}$-linear map

$$\eta_{\mathscr{M}} : \mathscr{M} \to U(\mathcal{F}(\mathscr{M}))$$

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- an $\mathscr{A}$-linear map

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such that every $\mathscr{A}$-linear map

$$\psi : \mathscr{M} \to U(\mathscr{B})$$

to an $\mathscr{A}$-algebra $\mathscr{B}$ factors uniquely through $\mathcal{F}(\mathscr{M})$:

$$\mathscr{M} \xrightarrow{\eta_{\mathscr{M}}} U(\mathcal{F}(\mathscr{M})) \xrightarrow{U(\widehat{\psi})} U(\mathscr{B}).$$

Relation with vector bundles

Theorem (Lerman–Louis)

Let $E \to M$ be a vector bundle. Then

$$\mathcal{F}(\Gamma(E)) \simeq C^\infty(E^*).$$

Moreover, the universal map

$$\eta_{\Gamma(E)} : \Gamma(E) \to C^\infty(E^*)$$

is the evaluation map

$$\alpha \mapsto \mathrm{ev}_\alpha\,.$$

Thus

$\mathcal{F}(\mathscr{M})$ is the correct replacement for the algebra of functions on the dual of a module.

Examples

Let M be a smooth manifold.

Example

- For derivations,

$$\mathcal{F}(\mathrm{Der}(C^\infty(M))) = C^\infty(T^*M).$$

- For Kähler differentials,

$$\mathcal{F}(\Omega^1(M)) = C^\infty(TM).$$

Examples

Let M be a smooth manifold.

Example

- For derivations,

$$\mathcal{F}(\mathrm{Der}(C^\infty(M))) = C^\infty(T^*M).$$

- For Kähler differentials,

$$\mathcal{F}(\Omega^1(M)) = C^\infty(TM).$$

Example (Tangent/cotengent bundle of a C^∞-ring)

For a C^∞-ring $\mathscr{A}$,

$$\mathcal{F}(\Omega_{\mathscr{A}})$$

should be its "tangent bundle" and $\mathcal{F}(\mathrm{Der}(\mathscr{A}))$ its "cotangent bundle".

Computing $\mathcal{F}(\mathscr{M})$

If $\mathscr{E}$ is a free $\mathscr{A}$-module with basis $(e_i)_{i\in I}$, then

$$\mathcal{F}(\mathscr{E}) \simeq \mathscr{A} \otimes_\infty C^\infty(\mathbb{R}^I).$$

If $\mathscr{E} \twoheadrightarrow \mathscr{M}$ is a surjective map of $\mathscr{A}$-modules, then

$$\mathcal{F}(\mathscr{M}) \simeq \mathcal{F}(\mathscr{E})/\langle \eta_{\mathscr{E}}(\ker(\mathscr{E} \to \mathscr{M}))\rangle.$$

Example

On the blackboard!

Linear Poisson brackets

Let $\mathscr{A}$ be a C^∞-ring and $\mathscr{M}$ an $\mathscr{A}$-module.

Definition

A Poisson bracket on $\mathcal{F}(\mathscr{M})$ is called linear if

$$\{\eta_{\mathscr{M}}(m_1), \eta_{\mathscr{M}}(m_2)\} \in \eta_{\mathscr{M}}(\mathscr{M}),$$

$$\{\eta_{\mathscr{M}}(m), \varphi_{\mathscr{M}}(a)\} \in \varphi_{\mathscr{M}}(\mathscr{A}),$$

and

$$\{\varphi_{\mathscr{M}}(a), \varphi_{\mathscr{M}}(b)\} = 0.$$

This is the analog of fiberwise linear Poisson structures on dual vector bundles.

Lie-Rinehart and Poisson C^∞-rings

Third main theorem

Theorem (Lerman–Louis)

Let $\mathscr{A}$ be a C^∞-ring and $\mathscr{M}$ an $\mathscr{A}$-module. The following data are equivalent:

1. *a Lie–Rinehart algebra structure on $\mathscr{M}$;*
2. *a linear Poisson C^∞-ring structure on $\mathcal{F}(\mathscr{M})$.*

The correspondence is determined by

$$\{\eta_{\mathscr{M}}(m_1), \eta_{\mathscr{M}}(m_2)\} = \eta_{\mathscr{M}}([m_1, m_2]_{\mathscr{M}}),$$

$$\{\eta_{\mathscr{M}}(m), \varphi_{\mathscr{M}}(a)\} = \varphi_{\mathscr{M}}(\rho(m)(a)),$$

$$\{\varphi_{\mathscr{M}}(a), \varphi_{\mathscr{M}}(b)\} = 0.$$

The proof is not straightforward!

The issue is that, for example, an element $u \in \mathcal{F}(\mathscr{M})$ may admit different realizations of the form

$$u = f_{\mathcal{F}(\mathscr{M})}(\eta_{\mathscr{M}}(m_1), \ldots, \eta_{\mathscr{M}}(m_n)) = g_{\mathcal{F}(\mathscr{M})}(\eta_{\mathscr{M}}(m'_1), \ldots, \eta_{\mathscr{M}}(m'_d)),$$

for different smooth functions $f\colon \mathbb{R}^n \to \mathbb{R}$, $g\colon \mathbb{R}^d \to \mathbb{R}$, and elements $m_i, m'_j \in \mathscr{M}$.

One must therefore make sure that the bracket is compatible with these relations.

Example: a singular quotient

Let

$$\mathscr{A} = \frac{C^\infty(\mathbb{R}^2)}{\langle xy \rangle}.$$

Then

$$C^\infty \mathrm{Der}(\mathscr{A}) = \langle \bar{x}\partial_x, \bar{y}\partial_y \rangle_{\mathscr{A}}.$$

The corresponding free algebra is

$$\mathcal{F}(C^\infty \mathrm{Der}(\mathscr{A})) \simeq \frac{C^\infty(\mathbb{R}^4)}{\langle xy,\ y\xi,\ x\nu \rangle}.$$

The Poisson bracket is induced by

$$\pi = x\,\partial_\xi \wedge \partial_x + y\,\partial_\nu \wedge \partial_y.$$

Merci de votre attention !

References



Eugene Lerman and Ruben Louis.

Lie-Rinehart and Poisson algebras over C^∞-ring.

arXiv preprint arXiv:2606.01388, 2026.