Topology & Geometry Seminar

IIIT Delhi

Sep 3rd, 2026

1. Motivations
2. Foliations & singular foliations
3. Lie-Rinehart algebras

} Category of manifolds ← Today!

4. C^∞-rings & Poisson C^∞-rings
5. Free "symmetric" C^∞-rings (the analog of the symmetric algebra)
6. Lie-Rinehart algebras $\simeq$ linear Poisson C^∞-rings over C^∞-rings

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① Overdetermined system of PDEs

$$\begin{cases} \dfrac{\partial f}{\partial x} = g(x,y) \\[2ex] \dfrac{\partial f}{\partial y} = h(x,y) \end{cases} \qquad (*)$$

where $g, h \in C^\infty(\mathbb{R}^2)$ and f unknown function.

- Here: Condition on g, h to have a solution:

$$\underbrace{\frac{\partial g}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial h}{\partial x}}$$

$$\boxed{\frac{\partial g}{\partial y} = \frac{\partial h}{\partial x}}$$

- Geometry: if $f : \mathbb{R}^2 \to \mathbb{R}$ is a (local) solution for $(*)$

$$\mathrm{Graph}(f) \hookrightarrow \mathbb{R}^3$$

is an embedded submanifold of $\mathbb{R}^3$.

- The tangent space at $(x, y, z) \in \mathrm{Graph}(f)$ is

$$\mathrm{Graph}(df_{(x,y)}) = \mathrm{span}\left(\begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y} \end{pmatrix} \right)$$

$$= \mathrm{span}\left(X(x,y,z), Y(x,y,z) \right)$$

$$X = \frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial}{\partial z}$$

Consider the module of vector fields $\mathcal{D} \subset \mathfrak{X}(\mathbb{R}^2)$ generated by X, Y.

$\mathfrak{X}(\mathbb{R}^2) = C^\infty(\mathbb{R}^2 \to \mathbb{R}^2) = \Gamma(T\mathbb{R}^2)$

Distribution

$$\underbrace{[X, Y]}_{\text{Lie bracket of v.f}} = X \circ Y - Y \circ X = \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y} \right) \frac{\partial}{\partial z}$$

$$\underbrace{[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}}_{\mathcal{D} \text{ involutive}} \iff \frac{\partial g}{\partial y} = \frac{\partial h}{\partial x}$$

For $(x,y,z) \in \mathbb{R}^3$, $\mathcal{D}_{(x,y,z)} = \text{Span}\left(\underline{X(x,y,z)}, \underline{Y(x,y,z)}\right)$ is of rank 2 : <u>Involutive</u> <u>Distribution</u> of <u>rank 2.</u>

Frobenius: $\mathcal{D}$ induces a $\overset{\text{foliation}}{\overline{\text{partition}}}$ of $\mathbb{R}^2$ by $\underbrace{\text{connected submanifolds}}_{\text{"leaves"}}$ of of dim=2.

Description of the leaves:

$$p: \mathbb{R}^3 \to \mathbb{R}^2, \quad (x,y,z) \to (x,y)$$

$$Tp(X) = \frac{\partial}{\partial x}, \quad Tp(Y) = \frac{\partial}{\partial y}$$

Inverse function theorem: p admits a local section

$i: \mathbb{R}^2 \to \mathbb{R}^3$ $(p \circ i = Id)$

$(x,y) \mapsto (x,y,f(x,y))$, where $f: \mathbb{R}^2 \to \mathbb{R}$ a smooth function.

$\Rightarrow$ a leaf $L_{(x_0,y_0,z_0)}$ is locally a graph of smooth function $f(x,y) = z$.

$\overset{\text{easily check}}{\Longrightarrow}$ $f(x,y) = z$ satisfies $\circledast$.

PDEs leads to foliations...

$\boxed{\text{Foliations are qualitative study of PDEs}}$

What is a (singular) foliation?

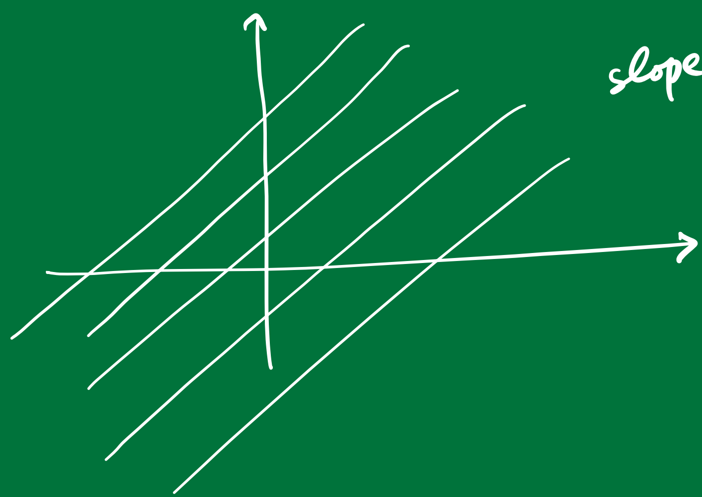
Partititionifold $\equiv$ Partition into leaves

Ex:

$\mathfrak{D} \subset \mathfrak{X}(\mathbb{R})$

$\mathfrak{D} = \text{span}(X)$

$X = \partial_x + \lambda \partial_y$



slope $= \lambda$

Parallel lines with slope $= \lambda$

when project to the Torus

$\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$

"Kronecker"

$= S^1 \times S^1$

The nature of the slope λ is crucial:

if $\lambda \in \mathbb{Q}$: Partition of $\mathbb{T}$ to circle

If not : lines wrapped around the torus in a dense way