Group of Self-Homotopy Equivalences

$\mathcal{C}$ - Category

$x \in ob(\mathcal{C})$.

$$Eq(x) := \{ f \in Mor_{\mathcal{C}}(x, x) \mid \exists g \in Mor_{\mathcal{C}}(x, x) \text{ s.t. } f \circ g = g \circ f = 1_x \}.$$

- This is a group under composition.

$\mathcal{C}$	$Eq(x)$
Groups	Automorphism group
Topological spaces	Homeomorphism group
Smooth manifolds	Diffeomorphism group.

$\mathcal{T}_h$ - objects :- Based Topological spaces

Morphisms :- Homotopy classes of based point preserving continuous maps.

$\mathcal{E}(X) := Eq(X) = \{ [f] : \exists g : (X, x_0) \to (X, x_0) \text{ s.t. } [f \circ g] = [g \circ f] = [I_X] \}$.

= collection of all homotopy classes of self homotopy equivalences on X.

objective :- Study $\mathcal{E}(X)$ for various $X \in ob(\mathcal{T}_h)$.

observations :- 1) $\mathcal{E}(X)$ is a homotopy invariant of X.

If $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ ($\simeq$), then $\mathcal{E}(X) \cong \mathcal{E}(Y)$

$[\gamma] \longmapsto [f \circ \gamma \circ g]$

2) This is not functorial

If $f: X \to Y$, then $\cancel{\rightsquigarrow}$ $\mathcal{E}(X) \rightleftarrows \mathcal{E}(Y)$

But if $T: \mathcal{T}_h \to \mathcal{C}$ is a functor, then.

$$T_* : \mathcal{E}(X) \longrightarrow Eq(TX) \quad \text{as} \quad T[f \circ g] = [Tf] \circ [Tg]$$

$$[f] \longmapsto [Tf]$$

$$T = \pi_* : \mathcal{T}_h \to \text{Groups}$$

$$H_* : \quad " \quad \to \quad "$$

$$H^* : \quad " \quad \to \quad "$$

k-th postnikov space : " $\to$ Top. *

$$\text{localization} : \quad " \quad \to \quad "$$

$$\Sigma : \quad " \quad \to \quad "$$

Literature : ▣ Barcus - Barratt ' 1958

K - 1-connected CW complex with cell dim $< q$.
and $X = K \underset{\alpha}{\cup} e^{q+1}$, $\alpha : S^q \to K$.

There is a exact sequence of three terms

$$i_*(\pi_{q+1}(K)) \longrightarrow \mathcal{E}(X) \to \begin{cases} \mathcal{E}(K) & \text{if } 2\alpha \neq 0 \\ \mathcal{E}(K) \oplus \mathcal{E}(S^{q+1}) & \text{if } 2\alpha = 0. \end{cases}$$

$i_* : \pi_{q+1}(K) \to \pi_{q+1}(X)$ induced by $i : K \to X$

▣ 1964' → Kahk, Shih, Arkowitz - Curjel.

* used postnikov decomposition of spaces to get various short exact sequences.

☒ 1966 - Nomura, Kudo - Tsuchida (1967), Rutter (1988), as so on.

<u>Properties :-</u>

- Any finite group can be realized as a subgroup of $\mathcal{E}(X)$ for some X

- G - finite group, $|G| = n$.

$G \hookrightarrow S_n$, symmetric group on n letters.

Y - non-contractible

$X = Y \times \cdots \times Y$ (n times).

$S_n \curvearrowright X$ by permuting the factors.

$S_n \longrightarrow \mathrm{Homeo}(X)$

As the permutation of factors of X is a homeomorphism,

$$S_n \longrightarrow \mathcal{E}(X) \qquad f_\sigma : X \to X$$
$$\sigma \longmapsto [f_\sigma] \qquad (x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

This is still injective.

i.e. if $\sigma \neq Id$, then $[f_\sigma] \neq [Id_X]$.

Because if $f_\sigma \simeq Id_X$, $\exists\, g_t : X \to X$ s.t. $g_0 = f_\sigma$, $g_1 = Id_X$

$\exists\, k, l$ s.t. $\sigma(k) = l$ with $k \neq l$ as $\sigma \neq Id$.

$$\begin{tikzcd}
Y \arrow[r, "H_t"] \arrow[d, "i_k"] & Y \\
Y \times \dots \times Y \arrow[r, "g_t"] & Y \times \dots \times Y \arrow[u, "Pr_l"]
\end{tikzcd}$$

$H_t :$ $y \mapsto (y_0, \dots, y, \dots, y_0)$ (k-th)

$y_0 \in Y$ fixed base pt.

Pr_l: Proj. onto l-th factor.

$\Rightarrow H_t = Pr_l \circ g_t \circ i_k$ and $H_0 = Id_Y$ and $H_1 = y_0$. cons.

$\Rightarrow Y$ is contractible

contradiction.

Examples / Basic computation :-

Ⓐ. $\mathcal{E}(S^n)$

$[f] \in \mathcal{E}(S^n)$. $\quad f: S^n \to S^n \rightsquigarrow f_*: \underset{\mathbb{Z}}{\underset{\cong}{H_n(S^n)}} \xrightarrow{\times d} \underset{\mathbb{Z}}{\underset{\cong}{H_n(S^n)}}$.

$$\deg(f) = d.$$

1) $f \sim g \iff \deg(f) = \deg(g)$

Hopf degree theorem.

2) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$ and $\deg(I_{S^n}) = 1$

If f is a homotopy equivalence, then $\deg(f) = \pm 1$.
and conversely.

$\Rightarrow \mathcal{E}(S^n) \cong \{\pm 1\} \cong \mathbb{Z}_2$.

(as $\deg(Id) = 1$ and $(x_0, \ldots, x_n) \mapsto (-x_0, \ldots, x_n)$ has degree -1.)

(B) $\mathcal{E}(K(G,n))$, $K(G,n)$: Eilenberg-Maclane spaces.

$$\pi_p(K(G,n)) \cong \begin{cases} G & \text{if } p = n \\ 0 & \text{otherwise.} \end{cases}$$

$f : K(G,n) \to K(G,n)$.

Recall :- $[X, K(G,n)] \longleftrightarrow H^n(X;G)$.

$X = K(G,n)$.

$$[f] \longleftrightarrow \alpha \in H^n(K(G,n);G)$$

$\cong$ $\leftarrow$ universal coefficient thm

$$\mathrm{Hom}_{\mathbb{Z}}(H_n(K(G,n)), G)$$

$\cong$ $\leftarrow$ Hurewicz

$$\mathrm{Hom}_{\mathbb{Z}}(G, G)$$

$\Rightarrow$ If f is a homotopy equivalence, then $[f] \longleftrightarrow \exists \in \mathrm{Aut}(G)$

Hence, $\mathcal{E}(K(G,n)) \cong \mathrm{Aut}(G)$.