2nd Talk : Computational techniques

Recall, $\mathcal{T}_h$ is category of based topological spaces with homotopy (htpy) class of base-point preserving continuous maps as morphisms.

Then for $X \in \mathcal{T}_h$,

$\mathcal{E}(X)$ is the group of self-htpy equivalences of X.

<u>Goal</u> :- Compute $\mathcal{E}(X)$

One of the fundamental tool for this is to use "Postnikov tower of X".

Postnikov tower :- X - connected space.

A Postnikov tower of X is a collection of spaces $\{X_n\}_{n \geq 0}$ with maps $\pi_{n+1} : X_{n+1} \to X_n$ & $P_n : X \to X_n$, $n \geq 0$ s.t.

a) $\pi_n : X_n \to X_{n-1}$ $(n \geq 1)$ is a principal fibration with fibre

$$K(\pi_n(X), n)$$

i.e., $\exists$ a fibration sequence $\Omega B' \to F' \to E' \to B'$ s.t. following diagram commutes upto htpy and vertical maps are htpy equivalences :-

$$\begin{tikzcd}
K(\pi_n(X), n) \arrow[r] \arrow[d, "\simeq"'] & X_n \arrow[r, "\pi_n"] \arrow[d, "\simeq"'] & X_{n-1} \arrow[d, "\simeq"', "q_{n-1}"] & \\
\Omega B' \arrow[r] & F' \arrow[r] & E' \arrow[r, "p'"'] & B'
\end{tikzcd}$$

Note that, as $\Omega B' \simeq K(\pi_n(X), n)$,

$$\pi_{n+1}(B') \cong \pi_n(\Omega B') \cong \pi_n(K(\pi_n(X), n)) = \pi_n(X)$$

and all other htpy groups of B' are zero

$\Rightarrow$ B' is Eilenberg-Maclane space $K(\pi_n(X), n+1)$

☑ Therefore for each $\pi_n : X_n \to X_{n-1}$, we get a map $k_{n+1} : X_{n-1} \to K(\pi_n(X), n+1)$ as $p' \circ q_{n-1}$ – this is unique upto htpy and thus its htpy class determines an element of $H^{n+1}(X_{n-1}, \pi_n(X))$ $\leftarrow$ this element is called (n+1)-th k-invariant of X and denoted by k_{n+1}.

b) $p_n : X \to X_n$ is an n-equivalence i.e.

$$\pi_k(p_n) : \pi_k(X) \to \pi_k(X_n) \text{ is an iso. for } k \leq n$$

c) $\pi_n \circ p_n \simeq p_{n-1}$

- This system, we will denote as $\{X_n, p_n, \pi_n\}$ and diagramatically :-

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_2 \arrow[d, "\pi_2"] \\
X \arrow[uur, dotted, no head] \arrow[ur, "p_2"] \arrow[r, "p_1"'] \arrow[dr, "p_0"'] & X_1 \arrow[d, "\pi_1"] \\
 & X_0
\end{tikzcd}$$

When does this system exist?

↳ If $\pi_1(X)$ acts on $\pi_n(X)$ trivially, then this system exists

obtained from the base point changing homomorphism

In particular, if X is simply connected, then it exists

Functoriality of this system

<u>Thm</u> [Kahn '63]: If X, X' have htpy type of 1-connected complexes and $f: X \to X'$. Then $\exists$ unique upto htpy $f_n: X_n \to X_n'$ for given Postnikov systems $\{X_n, p_n, \pi_n\}$ and $\{X_n', p_n', \pi_n'\}$ of X, X' resp. s.t.

$$\pi_n' \circ f_n = f_{n-1} \circ \pi_n, \quad p_n' \circ f \simeq f_n \circ p_n$$

and if k_n, k_n' are n-th k-invariants for X, X' resp.,

then $f_{n-2}^*(k_n') = f_\#^c(k_n)$, where

$$f_{n-2}^* : H^n(X_{n-2}' ; \pi_{n-1}(X')) \to H^n(X_{n-2} ; \pi_{n-1}(X'))$$

$$f_\#^c : H^n(X_{n-2} ; \pi_{n-1}(X)) \to H^n(X_{n-2} ; \pi_{n-1}(X'))$$

induced by the coeff. group homo.

$$\pi_{n-1}(f) : \pi_{n-1}(X) \to \pi_{n-1}(X').$$

Diagramatically,

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \\
& X_2 \arrow[r, "f_2"] \arrow[d, "\pi_2"] & X_2' \arrow[d, "\pi_2'"] & \\
X \arrow[ur, "p_2"] \arrow[r, "p_1"'] \arrow[dr, "p_0"'] \arrow[rrr, bend right=70, "f"'] & X_1 \arrow[r, "f_1"] \arrow[d, "\pi_1"] & X_1' \arrow[d, "\pi_1'"] & X' \arrow[ul, "p_2'"'] \arrow[l, "p_1'"] \arrow[dl, "p_0'"] \\
& X_0 \arrow[r, "f_0"] & X_0' &
\end{tikzcd}$$

With (htpy) commuting squares.

Observations :-

(a) $\{X_i\}, i<j$ is a postnikov system for X_j as follows -

$$
\begin{tikzcd}
 & \vdots \arrow[d, dotted] \\
 & X_j \arrow[d] \\
 & X_j \arrow[d, "\pi_j"] \\
 & X_{j-1} \arrow[d, "\pi_{j-1}"] \\
 & \vdots \arrow[d, "\pi_2"] \\
X_j \arrow[uuuur, dashed, "Id"] \arrow[uuur, "Id"] \arrow[uur, "\pi_j"'] \arrow[r, "\pi_2 \circ \cdots \circ \pi_j"'] \arrow[dr, "\pi_1 \circ \pi_2 \circ \cdots \circ \pi_j"'] & X_1 \arrow[d, "\pi_1"] \\
 & X_0
\end{tikzcd}
$$

all above spaces are X_j's

(b) There is a homomorphism $\mathcal{E}(X) \to \mathcal{E}(X_n)$ for each n

$\overbrace{}$ ← ignoring the class notation.

If $f \in \mathcal{E}(X)$, then $\exists g : X \to X$ s.t. $f \circ g = g \circ f \simeq \mathrm{Id}_X$.

Then by the functoriality of Postnikov system, we have $f_n : X_n \to X_n$ and $g_n : X_n \to X_n$ satisfying the above properties.

Then see that $f_n \circ g_n$ and $g_n \circ f_n$ will satisfies the required conditions for $f \circ g$ and $g \circ f$.

Thus by uniqueness of the maps, we have

$$f_n \circ g_n \simeq (f \circ g)_n \;\&\; g_n \circ f_n \simeq (g \circ f)_n$$

But $f \circ g \simeq \mathrm{Id} \simeq g \circ f \Rightarrow f_n \circ g_n \simeq \mathrm{Id}_{X_n} \simeq g_n \circ f_n$.

Thus $f_n \in \mathcal{E}(X_n)$ and therefore we have

map $\quad \mathcal{E}(X) \to \mathcal{E}(X_n)$

$$f \mapsto f_n$$

- This is also a homomorphism as we said above $(f \circ g)_n \simeq f_n \circ g_n$ satisfied.

(c) Combining (a) & (b), we have a homomorphism

$$\rho^n : \mathcal{E}(X_n) \to \mathcal{E}(X_{n-1})$$

[Kahn'64] Let X has Postnikov system (X_n, p_n, π_n). Then for $n \geq 3$, we have exact sequence

$$1 \to \ker(\rho^n) \hookrightarrow \mathcal{E}(X_n) \xrightarrow{\rho^n} \mathrm{Im}(\rho^n) \to 1.$$

Here exactness is immediate but the importance of the above sequence is that the $\ker(\rho^n)$ and $\operatorname{Im}(\rho^n)$ can be identified to other other subgroups:-

$$\ker(\rho^n) = \operatorname{Im}(\rho_F) \text{ and}$$

$$\operatorname{Im}(\rho^n) \cong \left\{ f_{n-1} \in \mathcal{E}(X_{n-1}) : \exists\, f^c \in \operatorname{Aut}(\pi_n(X)) \text{ s.t. } f^c_{\#}(k^{n+1}) = f^*_{n-1}(k^{n+1}) \right\}$$

Here $\rho_F : GF(X_n) \hookrightarrow \mathcal{E}(X_n)$ is the inclusion with

$$GF(X_n) := \left\{ f \in \mathcal{E}(X_n) : \begin{tikzcd} X_n \arrow[rr, "f"] \arrow[dr, "\pi_n"'] & \circlearrowleft & X_n \arrow[dl, "\pi_n"] \\ & X_{n-1} & \end{tikzcd} \right\}$$

Furthermore, $GF(X_n)$ is isomorphic as a set to a subset of $H^n(X_n\,;\,\pi_n(X))$

Above, $k^{n+1} \in H^{n+1}(X_{n-1}\,;\,\pi_n(X))$ is the $(n+1)$th k-invariant. and $f_{\#}^{c}$ is the usual coefficient homomorphism induced on H^{n+1} by $f^c \in Aut(\pi_n(X))$.

Using the above exact seq^n, Kahn deduced that

If X has finitely many non-zero finite homotopy groups or if X is a finite complex with finite homology groups then $\mathcal{E}(X)$ is finite.

Mapping cone method ← This is another useful method of calculating $\varepsilon(X)$, where X appears as mapping cone of some map.

- One such instance is when X is a closed, oriented smooth $(n-1)$-connected $2n$ manifold $(n \geq 2)$.

↳ For such X, it is known that X has the htpy type of the mapping cone of a map $b: S^{2n-1} \longrightarrow X_n$, X_n - bouquet of some (finite) S^n's. i.e. $X = X_n \cup_b CS^{2n-1}$

Then [Kahn '66] established the following exact seq^n:

$$[\Sigma X_n, X] \xrightarrow{(\Sigma b)^* + \psi} \pi_{2n}(X) \xrightarrow{K} \mathcal{E}(X) \xrightarrow{R} \mathcal{E}(X_n)$$

htpy class of maps from ΣX_n to X.

The description of the above maps are given as follows :-

$$(\Sigma b)^* : [\Sigma X_n, X] \to \pi_{2n}(X)$$

$$f \longmapsto \left[(\Sigma b)^*(f) : S^{2n} = \Sigma S^{2n-1} \xrightarrow{\Sigma b} \Sigma X_n \xrightarrow{f} X\right].$$

$$\psi : [\Sigma X_n, X] \longrightarrow \pi_{2n}(X)$$

$$\psi(f) = \sum_{k,l} \Gamma_{kl} [f \circ \Sigma e_k, i \circ e_l],$$ ← Whitehead product.

→ where $e_p : S^n \hookrightarrow X_n$ inclusions, determines classes in $\pi_n(X_n)$, inducing homology basis $\{x_1, \dots, x_r\}$, $r = \operatorname{rank} H_n(X) \geq 1$, of $H_n(X)$.

As $H^n(X) \cong \operatorname{Hom}(H_n(X), \mathbb{Z})$ here (by UCT),

$\{x_1^*, \dots x_r^*\}$ - dual basis of $H^n(X)$.

Then for each $1 \leq k, l \leq r$, we have

$$x_k^* \cup x_l^* = \Gamma_{kl}\, \nu^*,$$

cup product

ν^* is the dual of the orientation class $\nu \in H_{2n}(X) \cong \mathbb{Z}$.

→ $i : X_n \hookrightarrow X$ inclusion map

So, $\left(f \circ \Sigma e_k : S^{n+1} = \Sigma S^n \xrightarrow{\Sigma e_k} \Sigma X_n \xrightarrow{f} X\right) \in \pi_{n+1}(X)$

and $(i \circ e_1 : S^n \xrightarrow{e_1} X_n \xrightarrow{i} X) \in \pi_n(X)$.

$$\Rightarrow \quad [f \circ \Sigma e_K, i \circ e_1] \in \pi_{2n}(X).$$

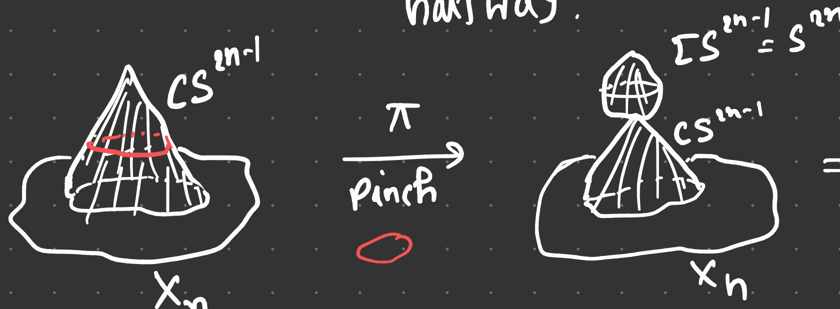
$K : \pi_{2n}(X) \longrightarrow \mathcal{E}(X)$.

$$K(f) : X \xrightarrow{\pi} X \vee S^{2n} \xrightarrow{1 \vee f} X \vee X \xrightarrow{\nabla} X,$$

folding map

$f \in \pi_{2n}(X)$

pinching the cone part halfway.

$\Big(X =$ 

$= X_n \vee S^{2n} \Big)$

$R : \mathcal{E}(X) \to \mathcal{E}(X_n)$

$R(f) : X_n \longrightarrow X_n$ is the htpy class in $X_n \to X_n$ of the restriction of a cellular representative of $f : X \to X$ to X_n

- This is well-defined by Whitehead's cellular approx. theorem.

- using the above exact sequence and the corresponding maps Kahn computed $\mathcal{E}(S^n \times S^n)$ for $n \geq 2$, even as semi-direct product.

Also, that $\mathcal{E}(\mathbb{C}P^2) \cong \mathbb{Z}_2 \cong \mathcal{E}(\mathbb{H}P^2)$