3rd Talk: Applications & Open Problems

Goal :- ① To describe applications of $\mathcal{E}(X)$, the group of (based) self homotopy equivalences of X.

② Direct to some open problems.

Applications

[A] Classification of Hurewicz fibration over circle.

Recall that ① $p : E \to B$ is a Hurewicz fibration with fibre F if for any top. space X, following com. diagram has a lift: $$\begin{tikzcd} X\times\{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ X\times[0,1] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$ that commutes and

$\exists$ a htpy equiv. $g : F \xrightarrow{\simeq} p^{-1}(*)$ for some $* \in B$.

② Two fibrations $p_i : E_i \to B$ with fibre F and htpy equiv. $g_i : F \to p_i^{-1}(*)$ are called <u>equivalent</u> if $\exists$ htpy equiv. $f : E_1 \to E_2$ such that $g_2 \simeq f \circ g_1$ and $p_1 \simeq p_2 \circ f$.

$$\begin{tikzcd} F \arrow[r, hook, "g_1"] \arrow[d, equal] & E_1 \arrow[r, "p_1"] \arrow[d, "f"] & B \arrow[d, equal] \\ F \arrow[r, "g_2"] & E_2 \arrow[r, "p_2"] & B \end{tikzcd}$$

- This is an equivalence relation.

Define $\mathcal{H}(B, F)$ to be the collection of equivalence classes of fibrations over B with fibre F.

[Allaud '66] There is a fibration $p_\infty : E_\infty \to B_\infty$ with htpy equiv. $g_\infty : F \to p_\infty^{-1}(*)$ s.t. there is a one-one correspondence :-

$$[B, B_\infty]_* \longleftrightarrow \mathcal{H}(B, F).$$

where $[B, B_\infty]_*$ is the collection of based htpy class of maps from B to B_∞.

⊠ Also, $\pi_i(B_\infty, *) \cong \pi_{i-1}(\mathcal{E}(F)_f, 1_F)$, where $\mathcal{E}(F)_f$ is the collection of all free self equivalences of F.

need not preserve base point

In particular, taking $B = S^1$, we see from the first corres. that $\mathcal{H}(S^1, F) \simeq [S^1, B_\infty]_* = \pi_1(B_\infty, *) \cong \pi_0(\mathcal{E}(F)_f, 1_F) \cong \mathcal{E}(F)_f$

Therefore, the equivalence class of fibrations over S^1 with fibre F corresponds to $\mathcal{E}(F)_f$, the group of htpy class of free self-equivalencies of F.

Note that, if $\pi_1(F)$ is trivial or F is an H-space, then $\mathcal{E}(F)_f \cong \mathcal{E}(F)$

Thus knowing the group $\mathcal{E}(F)$ in specific cases gives us the characterization of fibrations over S^1 upto equiv.

[B] Action of $\mathcal{E}(X) \times \mathcal{E}(Y)$ on $[X, Y]$.

Define $(\mathcal{E}(X) \times \mathcal{E}(Y)) \times [X, Y] \longrightarrow [X, Y]$ by

$$((\alpha, \beta), \varphi) \longmapsto \beta \circ \varphi \circ \alpha^{-1}$$

- Check that this is a well-defined action.

similarly, $\mathcal{E}(X)$ acts on $[X \times X, X]$ as well, as follows-

$$\mathcal{E}(X) \times [X \times X, X] \longrightarrow [X \times X, X]$$
$$(\alpha, \varphi) \longmapsto \alpha \circ \varphi \circ (\alpha^{-1} \times \alpha^{-1})$$

Note that, an H-space structure on X is a map $\mu : X \times X \to X$ such that there exists $e \in X$, $\mu(e,-), \mu(-,e) : X \to X$ are homotopic to Id_X.

Let $M(X) := \{[\mu] : \mu \text{ is an H-space structure on } X\} \subseteq [X \times X, X]$

$M_0(X) := \{[\mu] : \mu \text{ is a htpy associative H-space struc. on } X\}$
$\subseteq M(X)$.

The action of $\mathcal{E}(X)$ on $[X \times X, X]$ induces an action on $M(X)$ as well as on $M_0(X)$.

Take $\widetilde{M}(X) = M(X)/\mathcal{E}(X)$ and $\widetilde{M}_0(X) = M_0(X)/\mathcal{E}(X)$

[Curjel '68] For a finite, htpy. associative H-complex X, $\widetilde{M}_0(X)$ is always finite, and $\widetilde{M}(X)$ is infinite if and only if (the n-th Betti number number of $X \vee X$) $\times$ (rank of $\pi_n(X)$) $\neq 0$.

[C] Postnikov invariants:-

The above action also determines non-uniqueness of postnikov invariants.

(Adams '56) If X' is an n-section, $n \geq 2$ and G an abelian group, then the htpy types of $(n+1)$-sections with have n-type X' and $(n+1)$-th htpy group G is in 1-1 correspondence with

$$[X', K(G, n+2)] / \mathcal{E}(X') \times \mathrm{Aut}(G) \quad [\text{as } \mathcal{E}(K(G,n+2)) = \mathrm{Aut}(G)]$$

- A space X' is called an <u>n-section</u> if $\pi_i(X)=0$ for $i>n$.
- The <u>n-type</u> of X is the htpy type of any n-section X' s.t. $\exists$ a map $p: X \to X'$ with $p_*: \pi_i(X) \xrightarrow{\cong} \pi_i(X')$ for $i \leq n$

↪ one can always take X_n, the n-th space in Postnikov decom. of X, for X' as the n-type of X.

Thus from the above result, we have, the htpy type of X_{n+1}, the (n+1)-th Postnikov space of X, is uniquely determined by the htpy type of X_n, (n+1) th htpy group $\pi_{n+1}(X)$, and the equiv. class k^{n+1} of the postnikov invariant in $[X_n, K(\pi_{n+1}(X), n+2)] / \varepsilon(X_n) \times Aut(\pi_{n+1}(X))$

$$[X_n, K(\pi_{n+1}(X), n+2)] = H^{n+2}(X_n, \pi_{n+1}(X)) / \ldots$$

Therefore for uniqueness of Postnikov invariants, one should really consider those invariants as element of $H^{n+2}(X_n, \pi_{n+1}(X)) / \varepsilon(X_n) \times Aut(\pi_{n+1}(X))$

↳ this in particular determines obstruction classes for extending a map.

[D] Htpy actions of a group on a space :-

- A htpy action of a group G on a space X is a homomorphism

$$\alpha : G \to \mathcal{E}(X)_f$$

- If there exist such a map for a space X, then X is called htpy G- space.

- If for a htpy G- space X, the action map α factors through homeomorphism group, $\mathrm{Homeo}(X)$, of X, then X is called G- space. [i.e. X has an actual action of G]

Q. When a htpy G-action is realizable by a G-action?

[Cooke '78] A htpy G-action is realizable by a top. G-action iff $\exists$ a map $\theta : K(G,1) \to BE(X)_f$ s.t. following diagram is htpy commutative :-

$$\begin{tikzcd}
 & BE(X)_f \arrow[d, "B\pi"] \\
K(G,1) \arrow[ur, "\theta"] \arrow[r, "B\alpha"'] & K(\mathcal{E}(X)_f, 1)
\end{tikzcd}$$

Here $BE(X)_f$ is the classifying space for the monoid $E(X)_f$ via the Dold-Lashof construction.

$\alpha : G \to \mathcal{E}(X)_f$ is the htpy G-action map.

$\pi : E(X)_f \to \mathcal{E}(X)_f$ is the quotient map (quotienting by htpy).

This induces map on respective classifying spaces but $BG \simeq K(G,1)$ and $B\mathcal{E}(X)_f \simeq K(\mathcal{E}(X)_f, 1)$. so the above relati

Using above result, Zabrodsky '82 observed the following:-

Prop^n:- If X is a CW-complex, then $\exists$ a space Y of same htpy type as X such that every htpy class in $\mathcal{E}(X)_f$ contains a homeomorphism.

Open Problems

Q.1 a) Characterize those finite, connected complex X for which $\mathcal{E}(X)$ is finitely generated / finitely presented.

The Sullivan-Wilkerson theorem asserts that for simply connected finite complex X (i.e. a finite CW complex or has finitely many non-zero htpy groups), $\mathcal{E}(X)$ is finitely presented.

But for general finite complex $\mathcal{E}(X)$ is infinitely generated.

one such example was given by [Frank - Kahn '77].

b) Is there a finite, connected X, with $\mathcal{E}(X)$ finitely generated but not finitely presented.

As there are many more finitely gen. groups than finitely presented groups. so somewhat interesting to know such example.

c) Determine spaces X for which $\mathcal{E}(X)$ is trivial.

In general, for non-contractible space X, $\mathcal{E}(X)$ trivial is very rare

One such case is $K(\mathbb{Z}_2, n)$. as $\mathcal{E}(K(\mathbb{Z}_2, n)) \cong \mathrm{Aut}(\mathbb{Z}_2) = \{1\}$.

Kahn '76 also gave non-trivial example of spaces with trivial $\mathcal{E}(X)$.

so in a way, this is nice direction to pursue.

[a'] For simply connected finite CW-complex X, can we have the finite generation of the group $\mathcal{E}(\Omega X)$?

This is a question similar to (a) but for H-space ΩX, which is as a complex infinite one.

Also, a useful one in many places.

Q.2 Calculate $\mathcal{E}(X)$ for various well-known spaces, for example,

$X = \mathbb{H}P^n$ for $n > 2$

$X =$ any rank-2 H-spaces.

$X =$ compact Lie group, or Kahler manifold.

Q.3 For Moore spaces $M(G,n)$ $(n \geq 2)$, there is an epimorphism $\mathcal{E}(M(G,n)) \to \mathrm{Aut}(G)$ with kernel $\mathrm{Ext}(G, \pi_{n+1}(M(G,n)))$. Calculate the extension for $\mathcal{E}(M(G,n))$ precisely.