1. MOTIVATION. GROUPS AS METRIC SPACES

I. Introduction: Path-metrics and Cayley Graphs

$$(G, \cdot) \longleftrightarrow (G, d_S)$$

Algebraic object — Metric space

Algebraic features $\longleftrightarrow$ (Coarse) geometric features

- G f.g. group. Let S be a finite generating set. Assume S is symmetric ($s \in S \Rightarrow s^{-1} \in S$).

  $\Gamma = \mathrm{Cay}(G,S)$;
  - Vertex set of Γ is the underlying set of G
  - If $g \neq h \in G$, $\{g,h\}$ is an edge if $\exists s \in S$ such that $g = hs$.

- Given a simple (simplicial) graph $\Gamma = (\underset{\substack{\uparrow \\ \text{vertex set}}}{V}, \underset{\substack{\uparrow \\ \text{unordered pairs of vertices}}}{E})$ connected,

  $d_\Gamma : V \times V \to \mathbb{R}$;

  $(x,y) \mapsto d_\Gamma(x,y) = \min\{\operatorname{len}\gamma \mid \gamma \text{ is a path in } \Gamma \text{ from } x \text{ to } y\}$

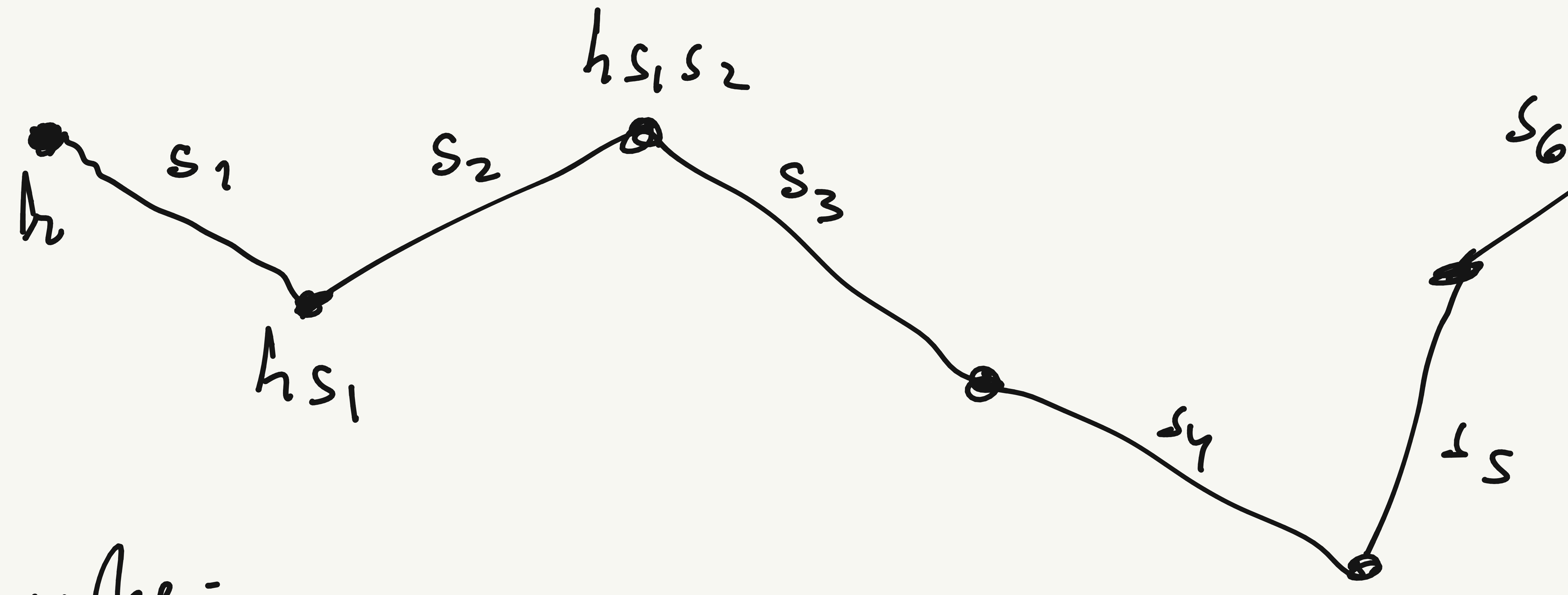
- Given a f.g. group G and a sym. fin. generating set S,

  $$(G,S) \underset{\text{conn. graph}}{\rightsquigarrow} \mathrm{Cay}(G,S) \rightsquigarrow (G, d_S)$$

  $d_S : G \times G \to \mathbb{R}$;

  $$d(g,h) = \underbrace{\|g^{-1}h\|}_{\text{word length}} =$$

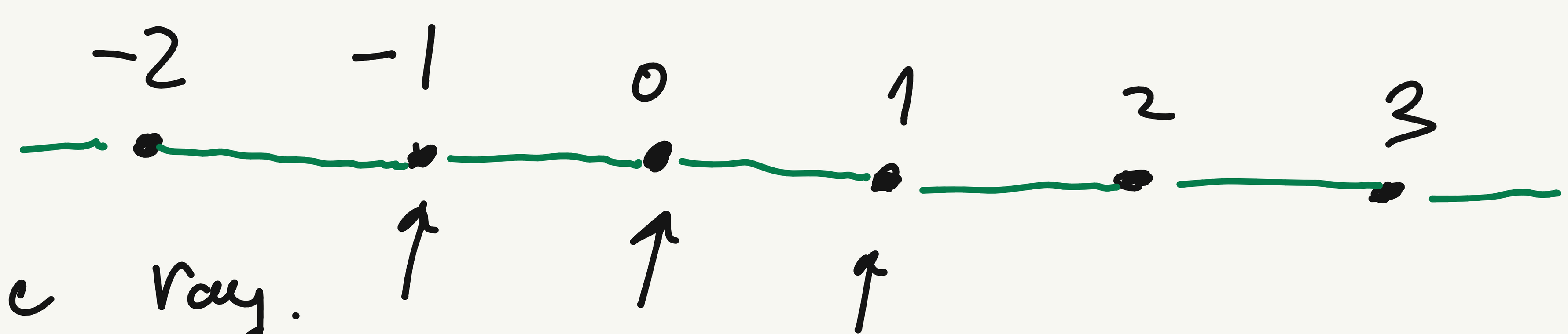
  $$= \min\{n \in \mathbb{Z}_{\geq 0} \mid \exists s_1, \dots, s_n \in S \text{ such that } g = h s_1 s_2 \cdots s_n\}$$



$g = h s_1 s_2 \cdots s_6$

Examples:

i) $\mathbb{Z} =: G$, $S = \{\pm 1\}$

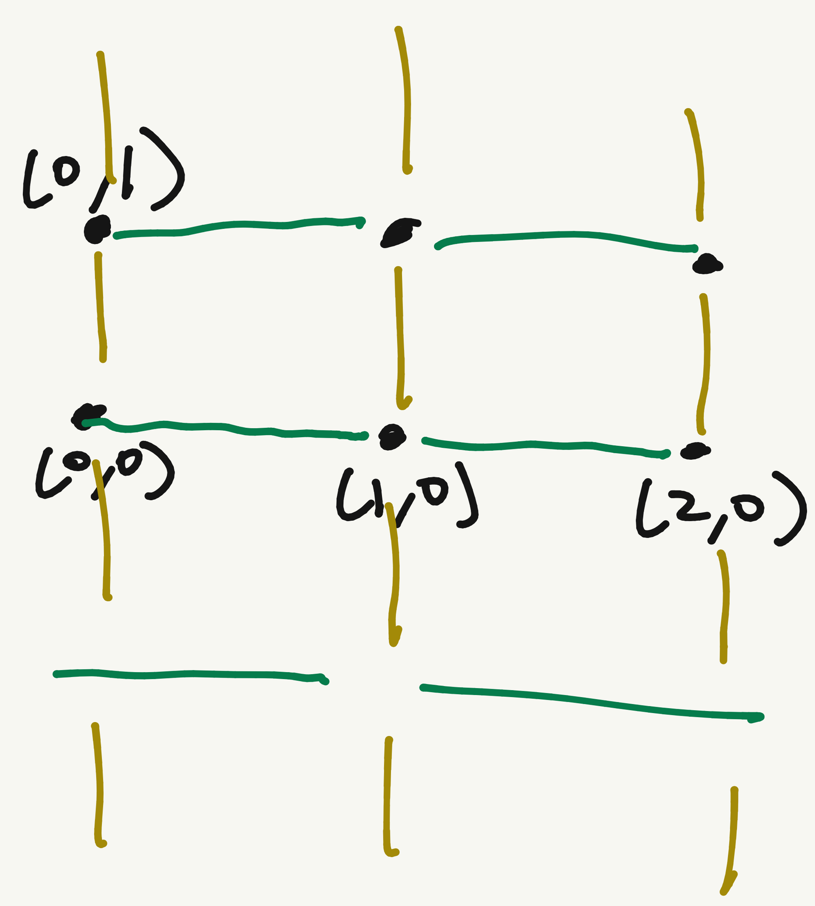


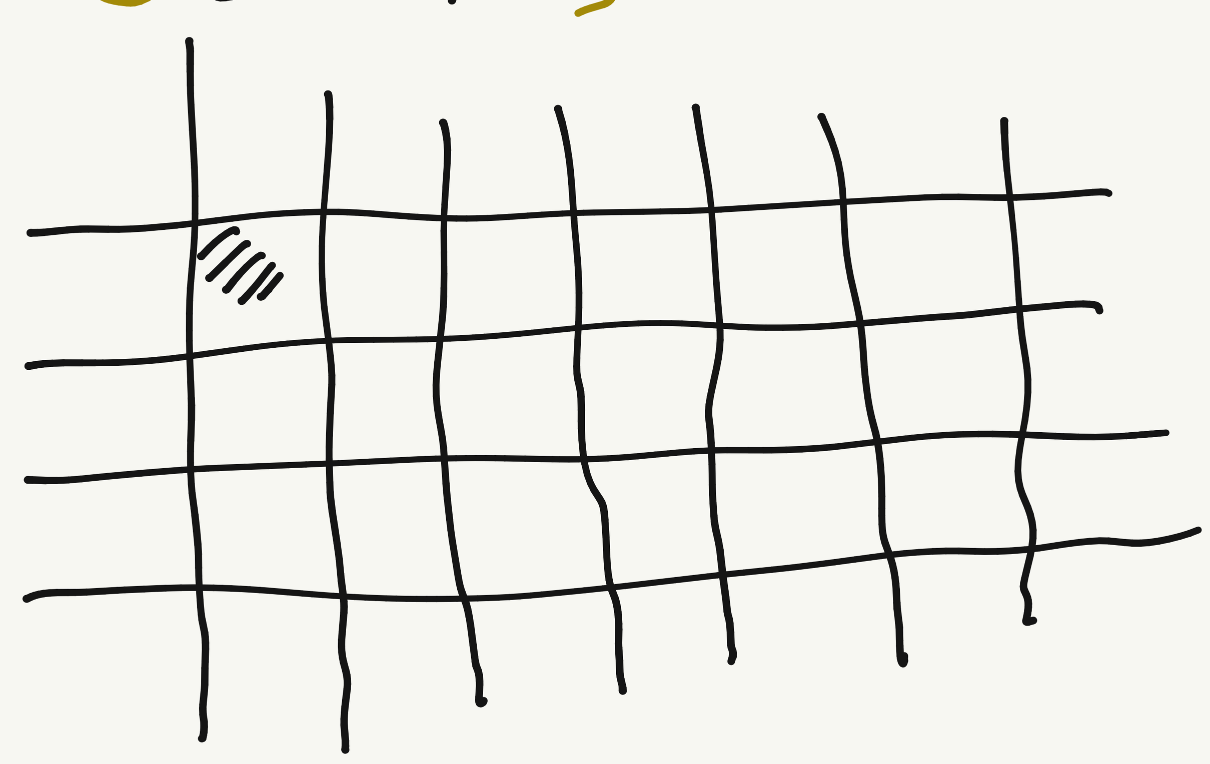
bi-infinite ray.

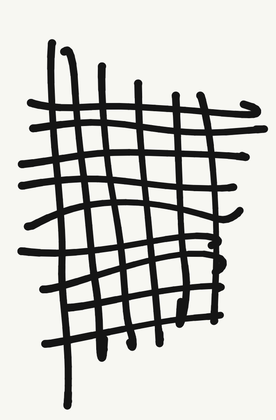
$\mathbb{Z} \hookrightarrow \mathbb{R}$;

$\mathbb{Z} \sim \mathbb{R}$

ii) $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$; $S = \{\boxed{\pm e_1}, \pm e_2\}$ $[e_1 = (1,0)]$ $[e_2 = (0,1)]$



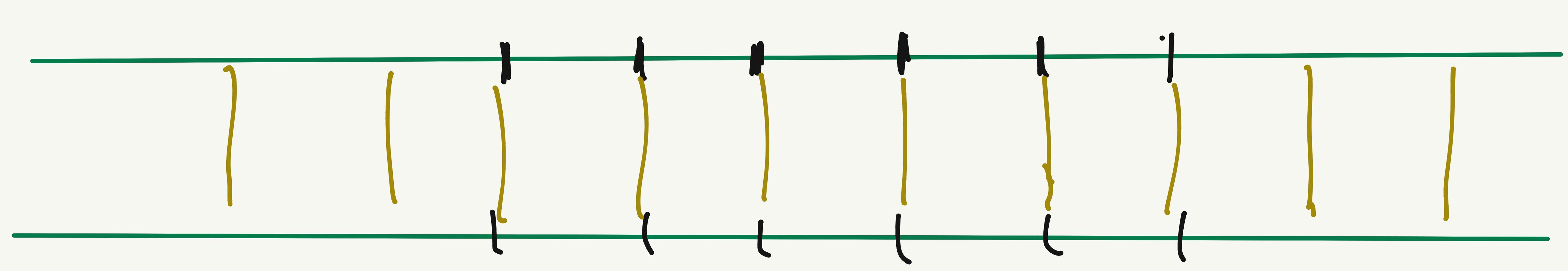




$\mathbb{Z}^2 \sim \mathbb{R}^2$

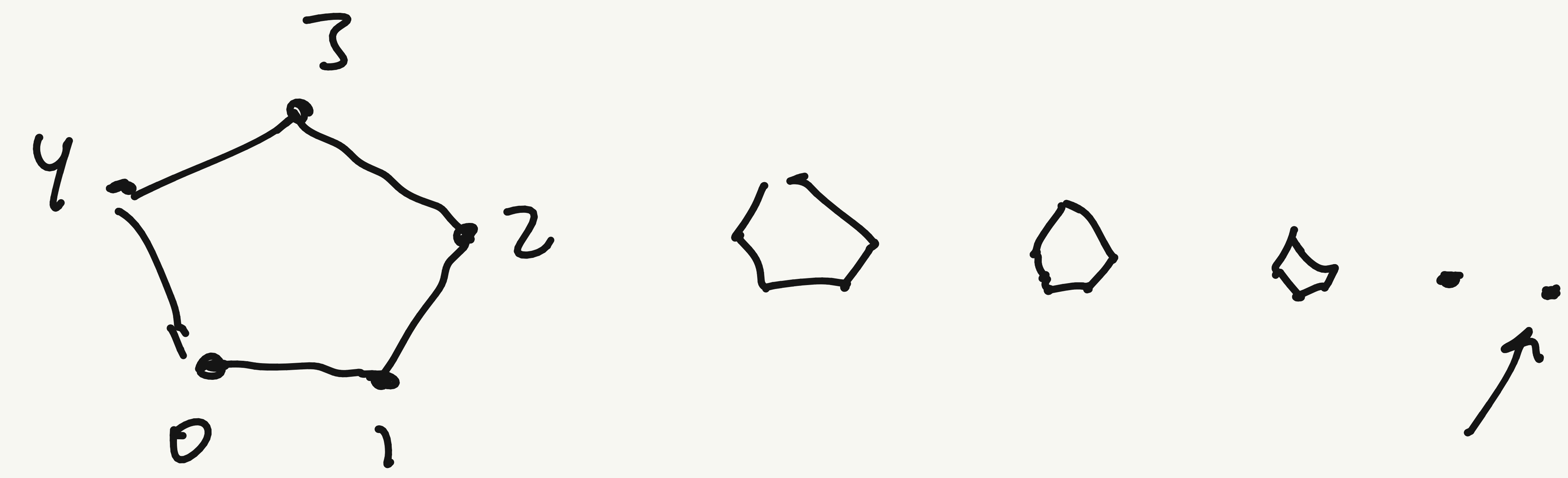
iii) $\mathbb{Z} \times \mathbb{Z}/2$; $S = \{\pm e_1, e_2\}$; $e_1 = (1,0)$, $e_2 = (0,1)$

$1+1=0$

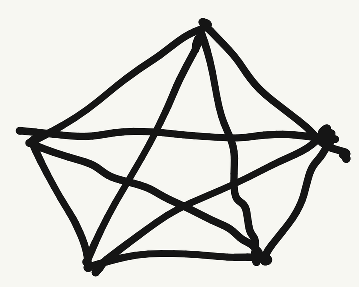


$\mathbb{Z}\times\mathbb{Z}/2 \sim \mathbb{R}$

iv) $\mathbb{Z}/5$, $S=\{\pm 1\}$;



$S=\{\pm 1, \pm 2\}$;

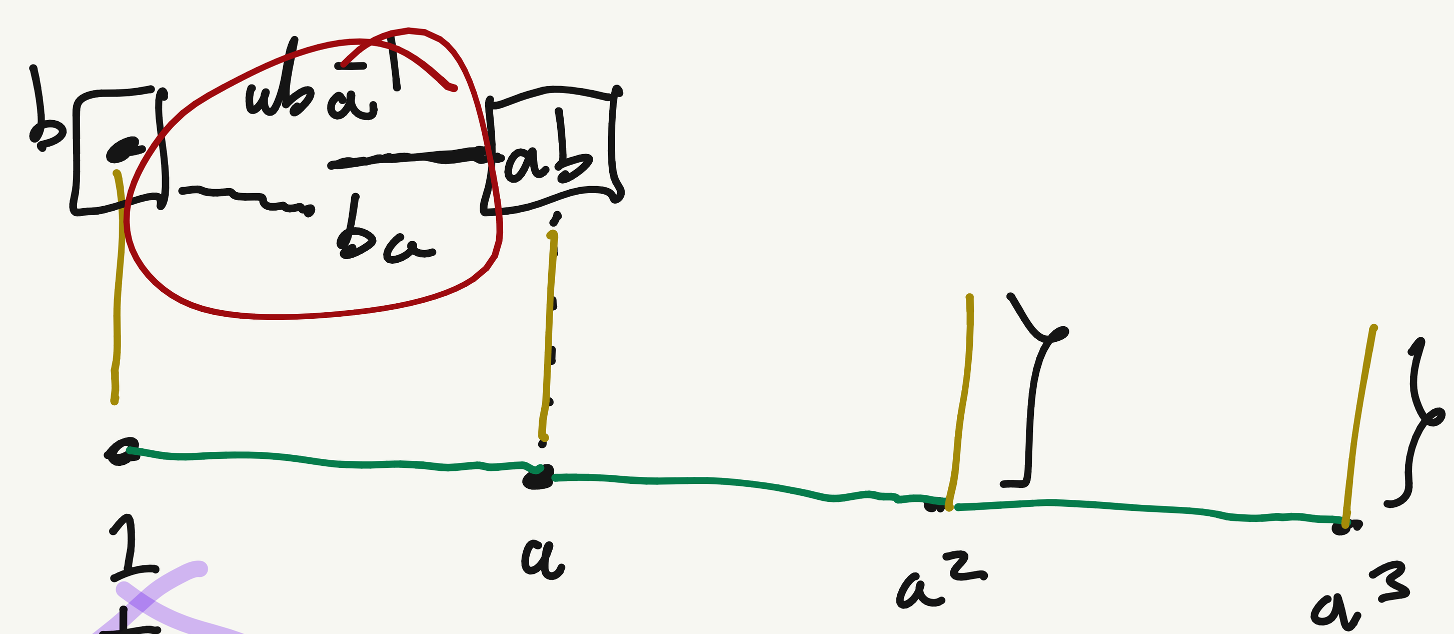


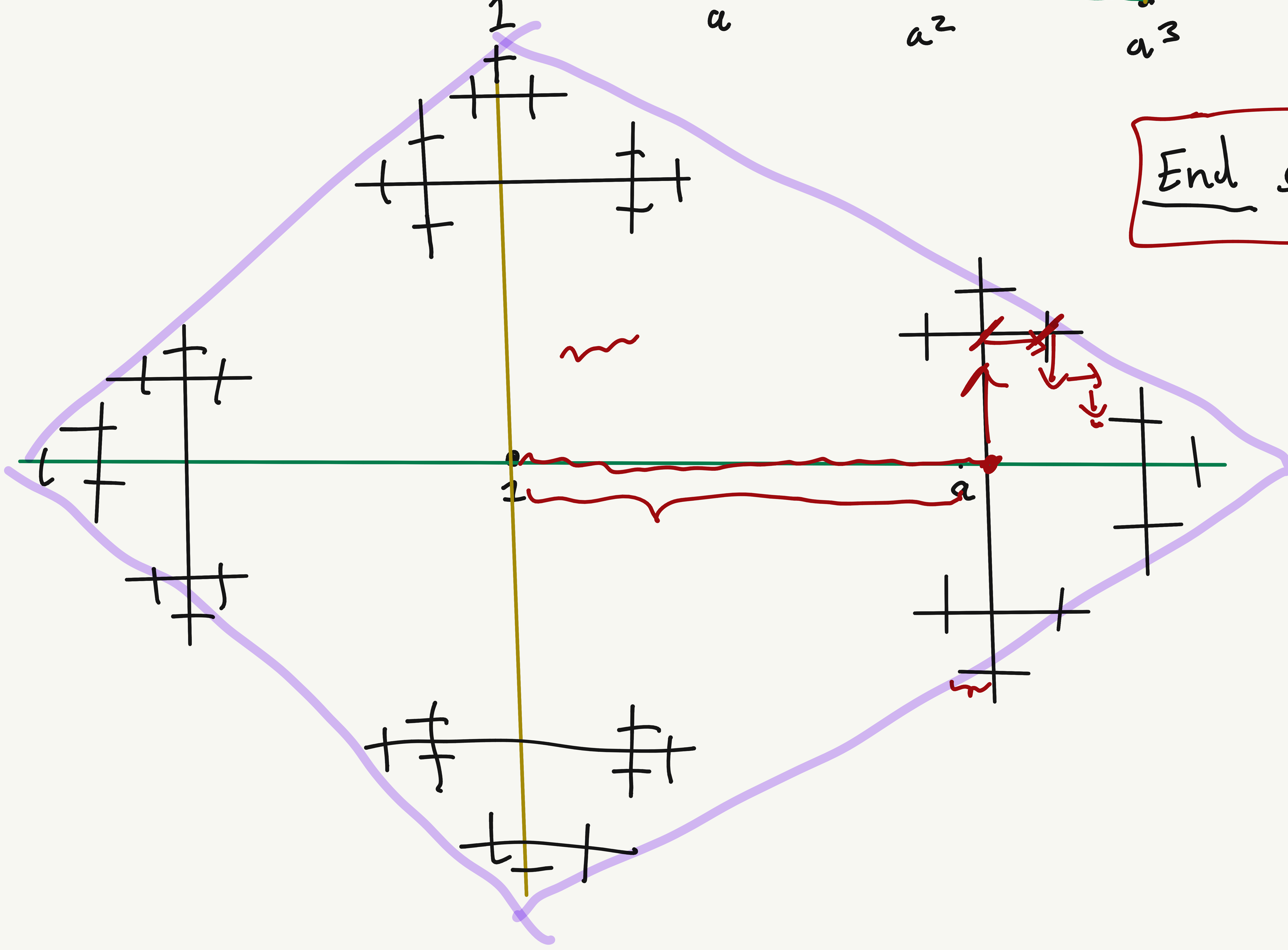
$\sim \cdot$

If G is finite, $G \sim *$.

v) $F_2 = \langle a, b\rangle = \mathbb{Z} * \mathbb{Z}$ $\quad S=\{\boxed{a^{\pm 1}}, \boxed{b^{\pm 1}}\}$

$\mathrm{Cay}(F_2, S)$





End space

II. Quasi-isometries

- Let $f:(X,d_X) \to (Y,d_Y)$ a map between metric spaces.

  i) Let $c>0$, $b\geq 0$. f is said to be a (c,b)-quasi-isometric embedding if

$$\frac{1}{c} d_X(x,x') - b \leq d_Y(f(x),f(x')) \leq c \cdot d_X(x,x') + b$$

  ii) f is a quasi-isometric embedding (q-i. emb.) if such c,b exist.

Let $f_1, f_2 : X \to Y$. f_1, f_2 are said to be "close" if $\exists r \geq 0$ s.t. $d(f_1(x), f_2(x)) \leq r$

- $f: X \to Y$ is a quasi-isometry (QI) if it is a q-i. emb and admits a q-i. emb. $g: Y \to X$ such that

$$gf \approx id_X \qquad fg \approx id_Y$$

$$\left(\begin{array}{llll} f: X \to Y \text{ in Top}, & gf \simeq id_X & \\ g: Y \to X & fg \simeq id_Y & \text{analogy} \end{array} \right)$$

- X and Y are quasi-isometric ($X \underset{QI}{\sim} Y$) if $\exists$ a QI between them.

* Examples:

i) $\mathbb{Z} =: G, \quad S = \{\pm 1\};$

$f : \mathbb{Z} \hookrightarrow \mathbb{R}$ is a QI. f is a $(1,0)$-q-i. emb.

$g : \mathbb{R} \longrightarrow \mathbb{Z}; \quad y \longmapsto \lfloor y \rfloor$ $\quad g$ is a $(1,2)$-q.-i. emb.

$gf = \mathrm{id}_{\mathbb{Z}} \approx \mathrm{id}_{\mathbb{Z}} \quad , \quad fg \approx \mathrm{id}_{\mathbb{R}}$

$$\mathbb{Z} \underset{QI}{\sim} \mathbb{R}$$

ii) $\mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$ is a QI. In general, $\mathbb{Z}^n \underset{QI}{\sim} \mathbb{R}^n$.

iii) $\mathbb{Z}/5 \underset{QI}{\sim} *$. In general, $G \underset{QI}{\sim} *$ iff G is finite.

Finiteness of f.g. is a QI (geometric) invariant.

iv) $G = \mathbb{Z} \times \mathbb{Z}/2$;

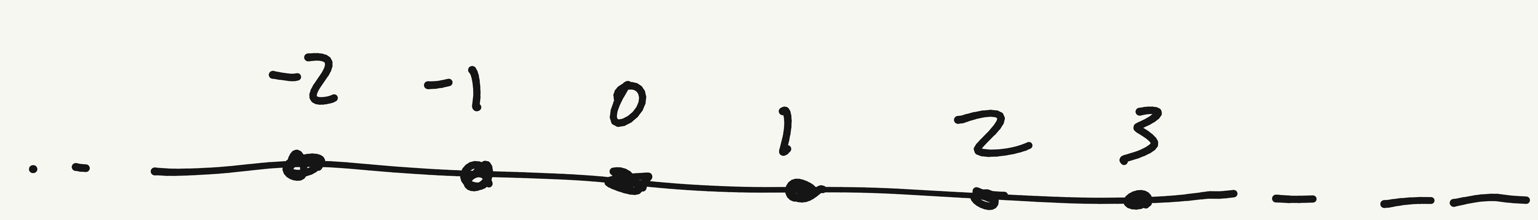
$$\begin{aligned} \mathbb{Z} &\xrightarrow{QI} \mathbb{Z} \times \mathbb{Z}/2 \\ x &\longmapsto (x, 0) \end{aligned}$$

v) $F_2 = \langle a, b \rangle$

* Dependance of $d_S : G \times G \longrightarrow \mathbb{R}$ on S.

$$G \text{ f.g} \dashrightarrow \mathrm{Cay}(G,S) \longrightarrow (G, d_S)$$

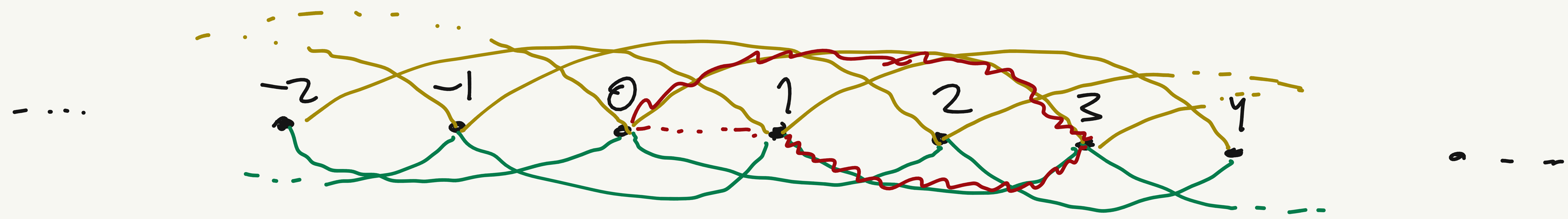
.) $\mathbb{Z} =: G \quad , \quad S = \{\pm 1\}$



$d_S(0,1) = 1$

.) $\mathbb{Z} =: G \quad , \quad S' = \{\pm 2, \pm 3\}$

•) $\mathbb{Z} =: G$, $S' = \{\boxed{\pm 2}, \boxed{\pm 3}\}$.



$$d_{S'}(0,3) = 1 \quad ; \quad d_{S'}(0,1) \underset{\substack{\le 2 \\ > 1}}{=} 2$$

▪ <u>Proposition</u>: Let G be a f.g. group and S, S' two finite symmetric generating sets. Then

$$f = \text{"}\mathrm{id}_G\text{"} : (G, d_{S'}) \longrightarrow (G, d_S)$$

is a QI.

<u>Proof</u>: Given $s' \in S'$, we can write $s' = s_1 \cdots s_{k(s')}$ for $s_1, s_2, \ldots, s_{k(s')} \in S$. Let $m_1 \in \mathbb{N}$ such that $k(s') \le m_1$ $\forall s' \in S'$.

Let $g \in G$. Consider $k = d_{S'}(1,g)$. $g = s_1' \cdots s_k'$ w/ $s_i' \in S'$.

$\longrightarrow$ g is written as a product of at most $k \cdot m_1$ elements of S.

$$d_S(1,g) \le m_1 \cdot d_{S'}(1,g). \quad \circledast$$

For arbitrary $g, h \in G$, $\boxed{d_S(g,h) \le m_1 \cdot d_{S'}(g,h)}$

$$d_S(g,h) = d_S(1, g^{-1}h) \quad \circledast$$

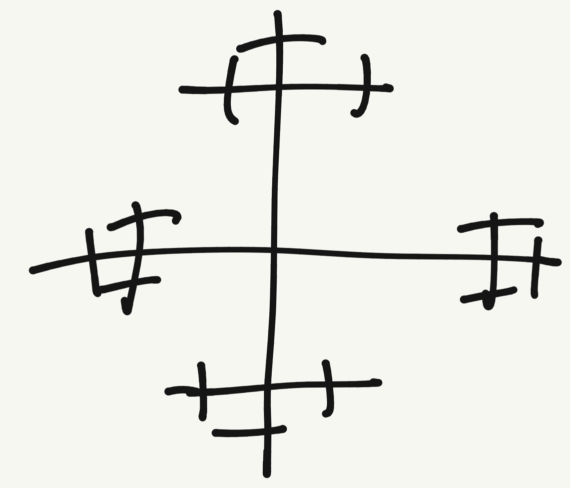
$$\left[\ \frac{1}{m_2} d_{S'}(g,h) \le d_S(g,h) \le m_1\, d_{S'}(g,h) \ \right]$$

If $m = \max\{m_1, m_2\}$, f is a $(m,0)$-q-i. emb.

f^{-1} is a q-i. emb. $\quad ff^{-1} = \mathrm{id} \approx \mathrm{id}$ $\quad$ f QI.

$\qquad f^{-1}f = \mathrm{id} \approx \mathrm{id}$ $\qquad \square$

- Free groups and their QI-type:

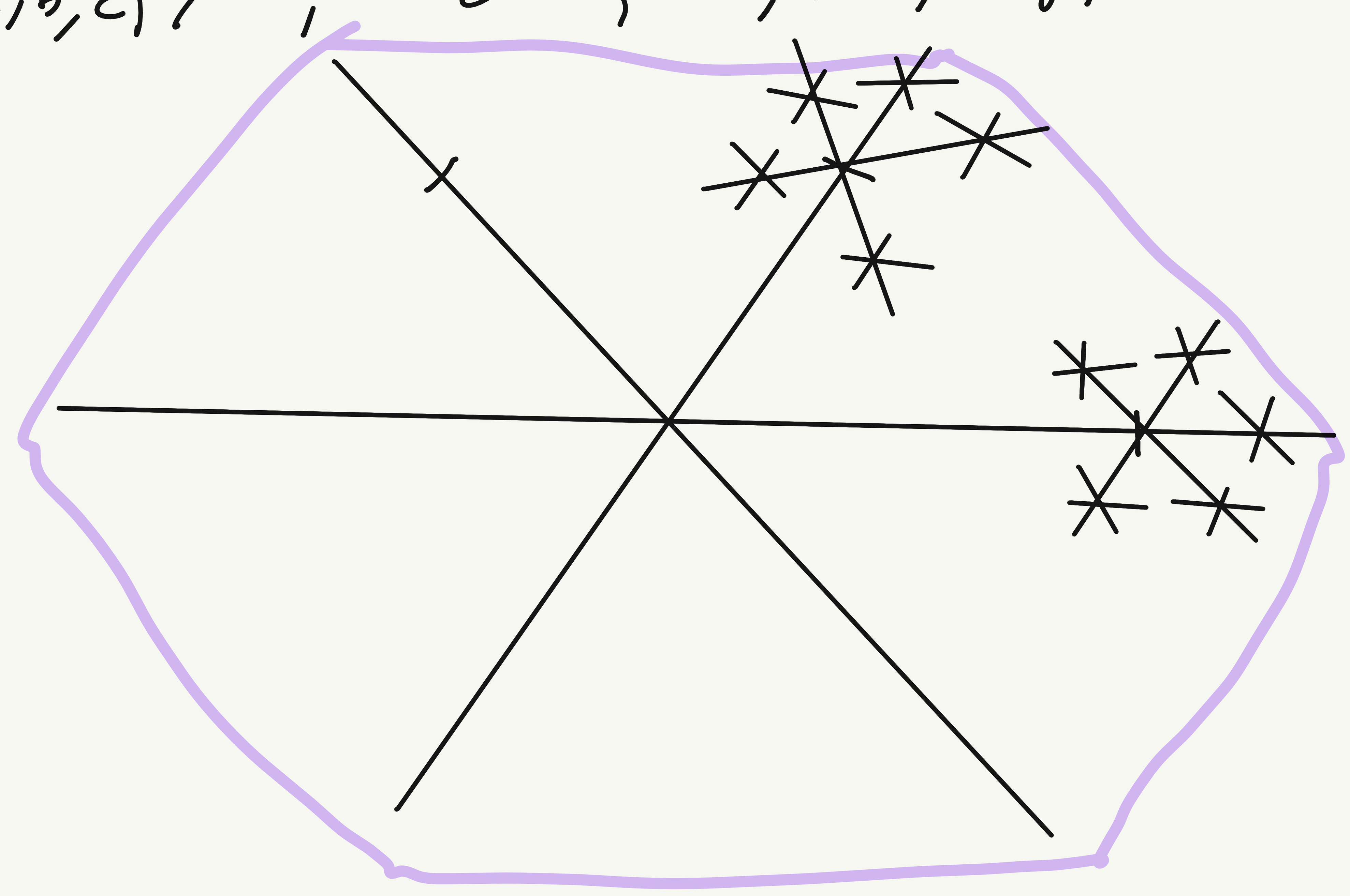


$F_1 = \langle a \mid \rangle = \mathbb{Z} \quad \underset{QI}{\not\sim} \quad F_2 = \langle a, b \mid \rangle$

$F_n = \langle x_1, \dots, x_n \mid \rangle = \mathbb{Z} * \overset{n}{\cdots} * \mathbb{Z} \;;\; n \geq 2$

$F_1 \underset{QI}{\not\sim} \left[F_2 \underset{QI}{\sim} F_n \right] \; \forall n \geq 2$

$F_3 = \langle a, b, c \mid \rangle \;;\quad S = \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\} \;;$



Rank of a group G is <u>not</u> a geometric property.
(min. number of generators a fin. gen. set admits)

Algebraic property

<u>Q</u>

- Consider the group $G = \mathrm{Isom}(\mathbb{Z})$ of isometries of $\mathbb{Z}$.
  Find a finite gen. set S of G, draw <u>$\mathrm{Cay}(G,S)$</u>
  and recognize its quasi-isometry type.

  $G = D_\infty$ infinite dihedral group.