# II. The coarse category. Coarse Cohomology

## 2.0. Reminder:

- $(G,\cdot)$ f.g. group $\underset{\text{s. fin. gen. set}}{\longleftrightarrow}$ $\boxed{\text{Cay}(G,S) \text{ connected graph}}$, $\boxed{(G,d_S) \text{ metric space}}$

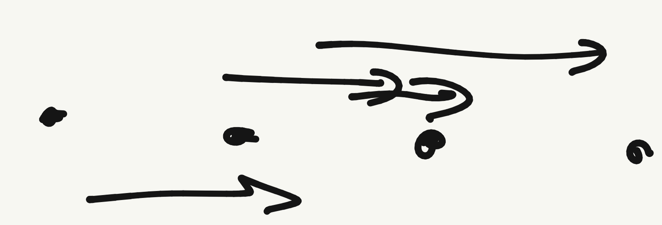
  Algebra - - - - - - - - - Geometry at infinity

- Quasi-isometries: $X \overset{QI}{\sim} Y$ if $X,Y$ look the same from far away.
- From far away, the way $\boxed{(G,d_S)}$ looks does not depend on $S$:

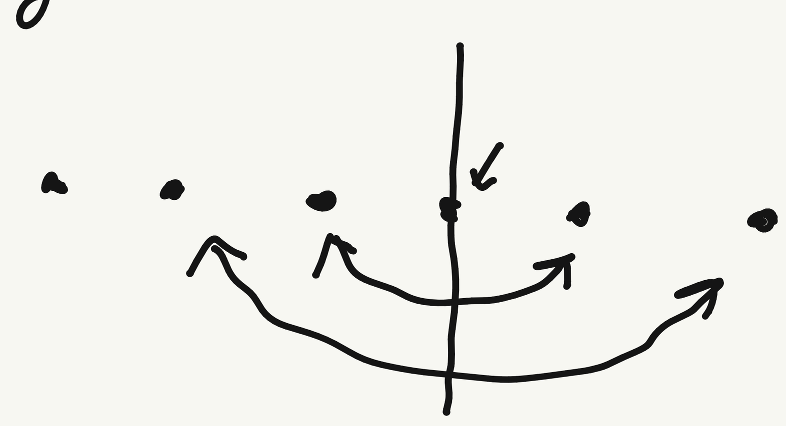
  [ $S,S'$ fin. gen. sets $\Rightarrow$ $id_G : (G,d_{S'}) \xrightarrow{QI} (G,d_S)$. ]

- Pending exercise: Recognize $Isom(\mathbb{Z}) = D_\infty$ from far away.

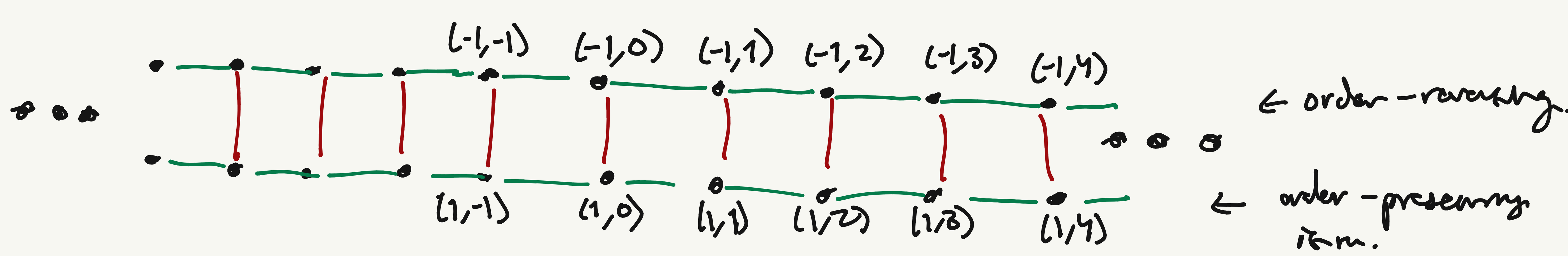
  Two kinds of isom. order-preserving: translations $x \mapsto x+b$

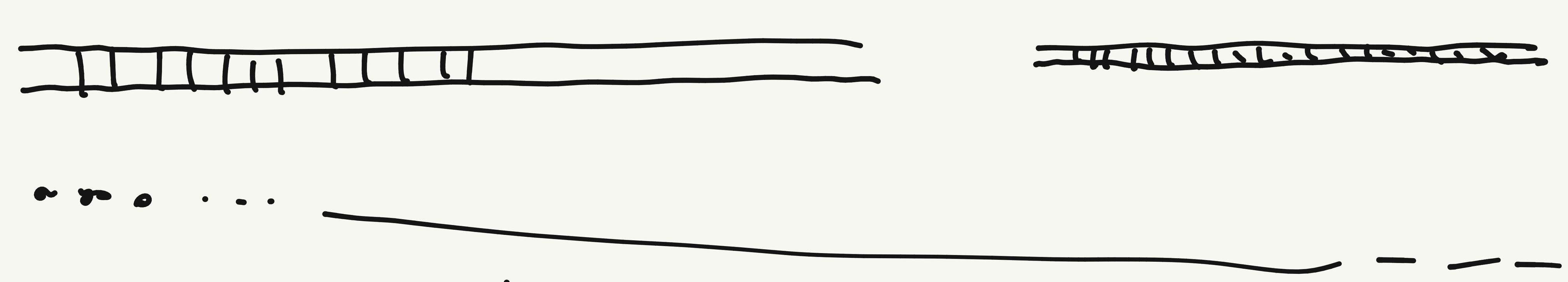


  order-reversing: symmetries



$Isom(\mathbb{Z}) =$
$= D_\infty = \langle t, s \rangle$
$\boxed{t} = (x \mapsto x+1)$
$\boxed{s} = (x \mapsto -x)$



$G = D_\infty$, $S = \{t^{\pm 1}, s\}$; $\text{Cay}(G,S) \underset{QI}{\sim} \mathbb{R} \underset{QI}{\sim} \mathbb{Z}$.

$\mathrm{Isom}(\mathbb{Z}) = \{ f_{a,b} : \mathbb{Z} \to \mathbb{Z},\ x \mapsto ax + b \mid a \in \{\pm 1\},\ b \in \mathbb{Z} \}$

$f_{a,b} \sim (a,b)$. $\qquad f_{1,0} : x \mapsto x$

$f_{a',b'} f_{a,b}(x) = a'(ax+b) + b'$;

$(a',b')(a,b) = (a'a,\ a'b + b')$.

## 2.1. Schwarz-Milnor lemma

- Let $G$ be a group acting by isometries on a proper geodesic metric space $(X,d)$. Assume $G \curvearrowright X$ is:
  - i) proper: $\forall K \subseteq X$ compact, $\{ g \in G \mid gK \cap K \neq \emptyset \}$ is finite.
  - ) cocompact; $X/G$ is compact.

  Then $G$ is f.g. and $G \underset{QI}{\sim} X$. Moreover, for every $x_0 \in X$,

$$\begin{aligned} f : G &\longrightarrow X \\ g &\longmapsto g \cdot x_0 \end{aligned}$$

is a quasi-isometry.

- $(M,g)$ closed connected Riem. manifold.

  Let $p : \tilde{M} \to M$ local. isometry. $\tilde{M}$ is a Riem. structure.

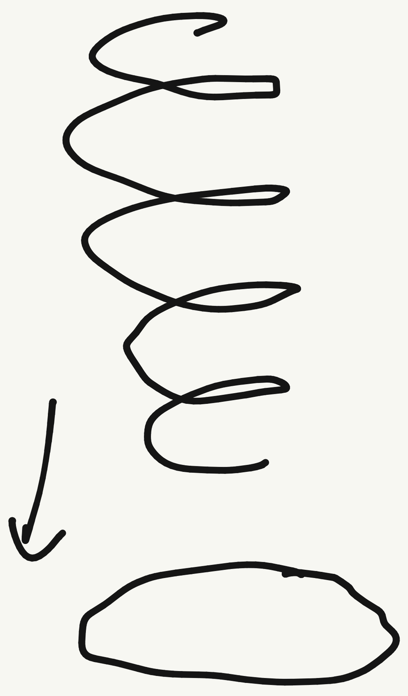
  $\pi_1(M) \curvearrowright \tilde{M}$ satisfies the hypothesis above. Then

  ) $\pi_1(M)$ is finitely generated.

  ) $\pi_1(M) \underset{QI}{\sim} \tilde{M}$.

- $M = \mathbb{S}^1$ closed Riem. mnfld.

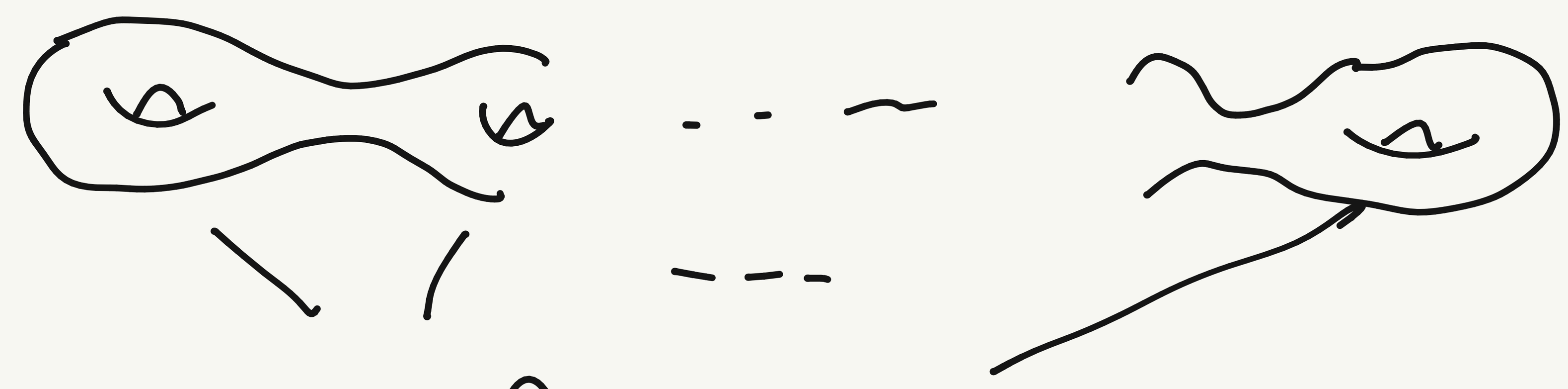
  $p : \underbrace{\mathbb{R}}_{\sim} \longrightarrow \underbrace{\mathbb{S}^1}_{\sim}$ ; By Schwarz – Milnor,



$p$

$$\underbrace{\pi_1(\mathbb{S}^1)}_{\mathbb{Z}} \underset{QI}{\sim} \mathbb{R}$$

- $M$ is a closed surface, orientable of genus $g \geq 2$.

$M =$ 

$g$

$$\widetilde{M} = \mathbb{H}^2 \qquad \pi_1(M) \underset{QI}{\sim} \mathbb{H}^2.$$

- $M = \mathbb{S}^1 \times \mathbb{S}^1$ ; $\widetilde{M} = \mathbb{R}^2$ ; $\underbrace{\pi_1(\mathbb{S}^1 \times \mathbb{S}^1)}_{\mathbb{Z} \times \mathbb{Z}} \underset{QI}{\sim} \mathbb{R}^2$.

## 2.2. coarse structures.

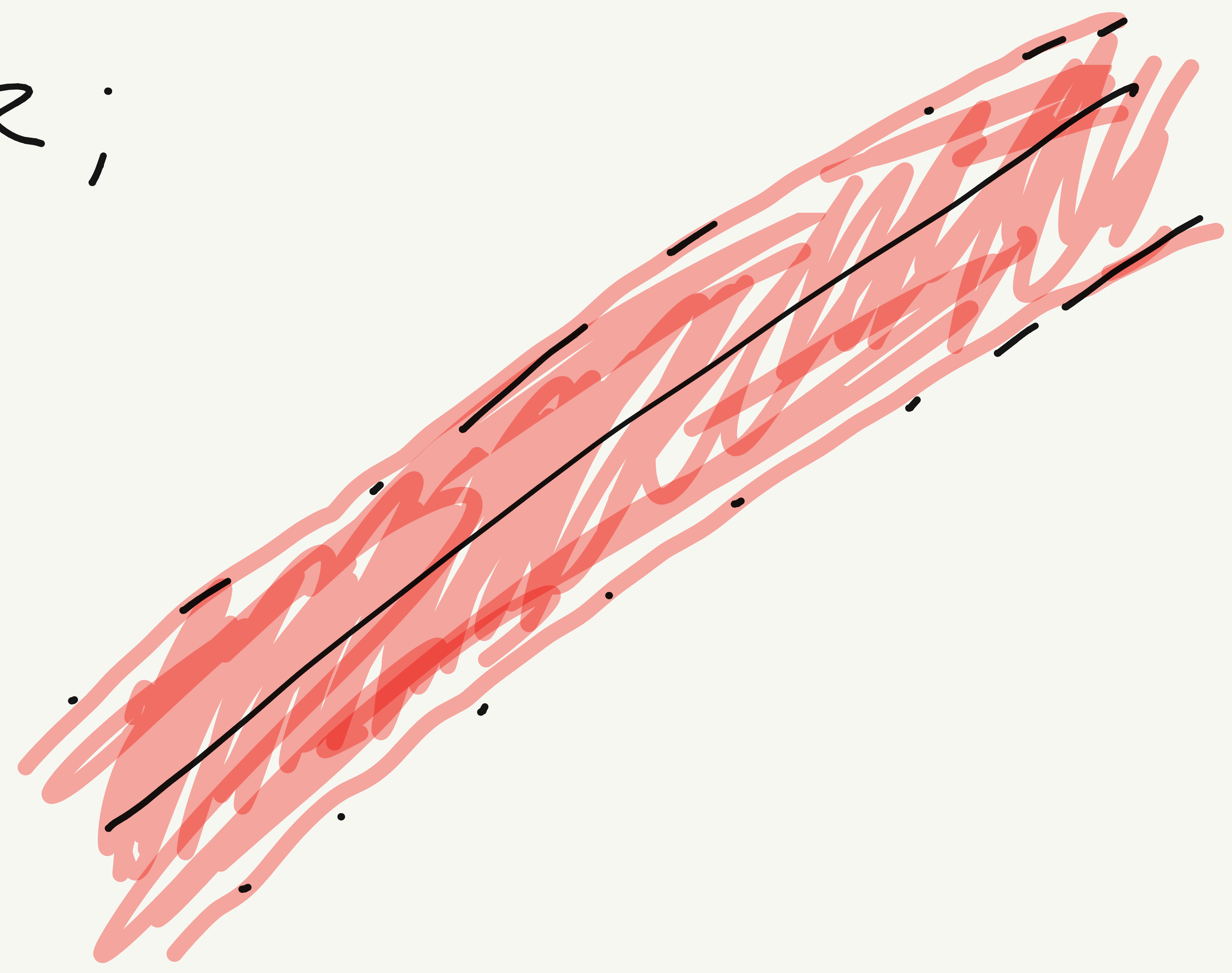
- $G \;\cancel{\dashrightarrow}\; (G, d_S) \longrightarrow \boxed{(G, \mathcal{E})}$ ¿?

  $\mathcal{E}$ structure encoding the large scale geometry

• $(X,d)$ metric space. For each $r>0$,

$$T_r := \{(x_1,x_2) \in X \times X \mid d(x_1,x_2) \le r\}$$

If $X = \mathbb{R}$;



i) $s \le r$, $T_s \subseteq T_r$.

ii) $r_1, r_2 > 0$, $T_{r_1} \cup T_{r_2} \overset{(=)}{\subseteq} T_{\max\{r_1,r_2\}}$

iii) $\Delta \subseteq T_r \quad \forall r > 0$.

iv) (Triangle inequality) $\quad T_{r_1} \overset{?}{;} T_{r_2} \subseteq T_{r_1+r_2}$

Given $x_1, x_2, x_3 \in X$, $\quad (x_1,x_2);(x_2,x_3) = (x_1,x_3)$

$x_1 \to x_2 \longrightarrow x_3$

$$T_{r_1};T_{r_2} = \left\{(x_1,x_3) \;\middle|\; \exists x_2 \in X \text{ s.t. } \begin{array}{l} d(x_1,x_2) \le r_1 \\ d(x_2,x_3) \le r_2 \end{array}\right\}$$

$d(x_1,x_3) \le r_1 + r_2$.

v) If $(x_1,x_2) \in X \times X$, $\quad (x_1,x_2)^{-1} = (x_2,x_1)$.

If $E \subseteq X \times X$, $\quad E^{-1} = \{(x_2,x_1) \mid (x_1,x_2) \in E\}$.

$T_r = T_r^{-1}$.

- <u>Def'n</u>: Let $X$ be a set. A <u>coarse structure</u> on $X$ is a family $\mathcal{E}$ of subsets of $X \times X$ ("entourages" or "controlled sets") satisfying:

  i) $E_1 \subseteq E_2 \in \mathcal{E} \Rightarrow E_1 \in \mathcal{E}$.

  ii) $E_1, E_2 \in \mathcal{E} \Rightarrow E_1 \cup E_2 \in \mathcal{E}$.

  iii) $\Delta_X = \{(x,x) \mid x \in X\} \in \mathcal{E}$.

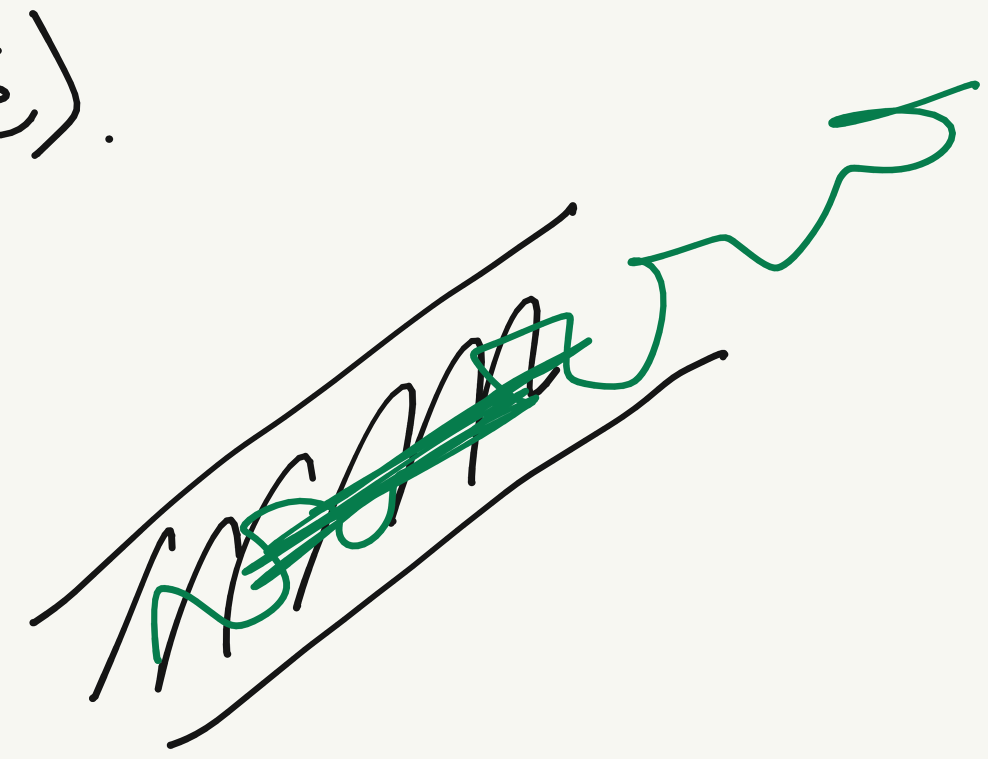
  iv) $E_1, E_2 \in \mathcal{E}$,

  $$\Rightarrow E_1 ; E_2 = \left\{ (x_1, x_3) \;\middle|\; \exists x_2 \in X \text{ s.t. } \begin{matrix} (x_1,x_2) \in E_1 \\ (x_2,x_3) \in E_2 \end{matrix} \right\} \in \mathcal{E}$$

  $$\begin{tikzcd} x_1 \arrow[r] \arrow[rr, bend right] & x_2 \arrow[r] & x_3 \end{tikzcd}$$

  v) $E \in \mathcal{E} \Rightarrow E^{-1} \in \mathcal{E}$.

A <u>coarse space</u> is a pair $(X, \mathcal{E})$.



## 2.3. Examples

- (Bdd) metric coarse structure:

  Let $(X,d)$ be a metric space. Then

  $$\mathcal{E}_d := \{ E \subseteq X \times X \mid \exists r > 0 \text{ s.t. } \underbrace{E \subseteq T_r}_{d(x_1,x_2) \le r \;\; \forall (x_1,x_2) \in E} \}$$

  $$= \{ E \subseteq X \times X \mid \sup_{(x_1,x_2) \in E} d(x_1,x_2) < \infty \}$$

  $$\underset{\text{f.g.}}{(G, \cdot)} \dashrightarrow \underset{\substack{\text{metric}\\ \text{space}}}{(G, d_S)} \xrightarrow[\text{metric structure}]{\text{Bdd}} \underset{\substack{\text{coarse}\\ \text{space}}}{(G, \mathcal{E})}$$

- $C_0$ - metric coarse structure.

$(X,d)$ metric space. Let $E \subseteq X^2$. Then,

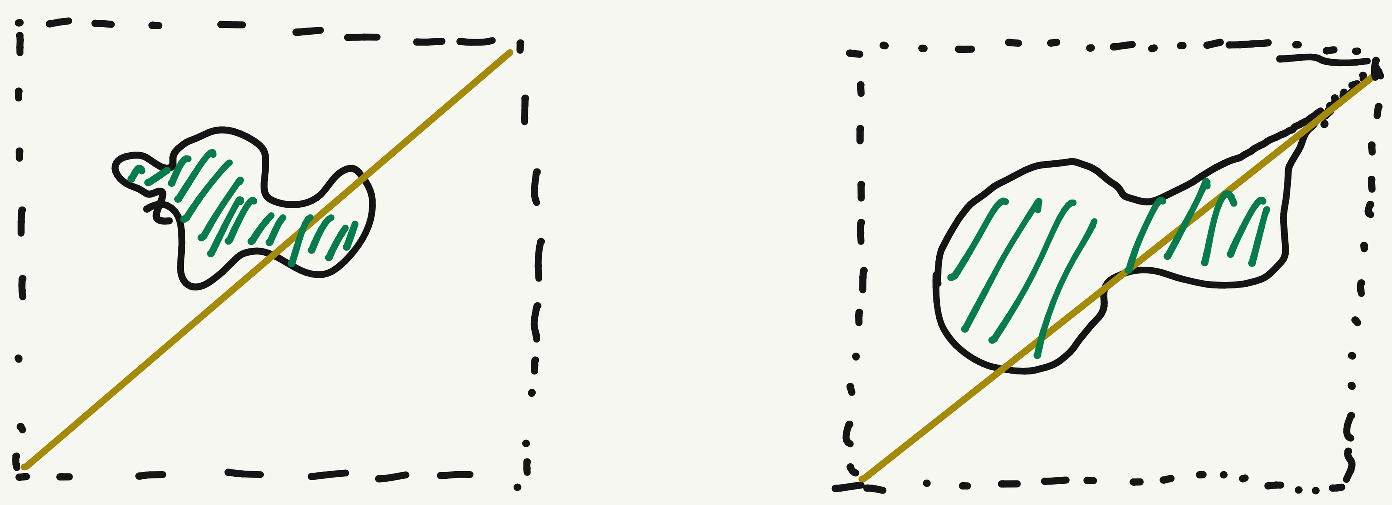
$E \in \mathcal{E}_{C_0} \iff d|_E : E \to \mathbb{R}$ "tends to zero at infinity"

$\iff \forall \varepsilon > 0 \ \exists K \subseteq X$ cpt such that

$d|_{(E \setminus K \times K)} < \varepsilon$.

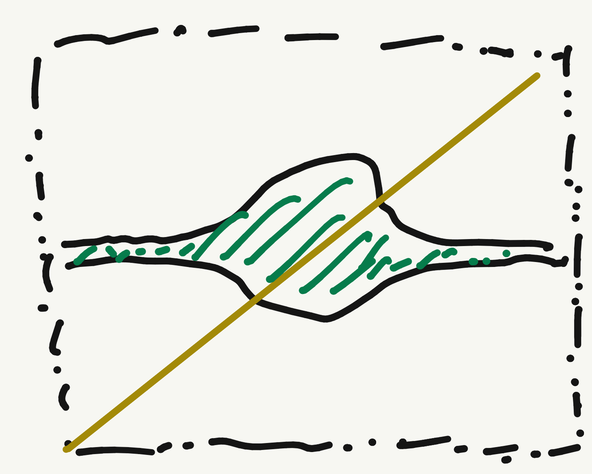
$\Big((x_1, x_2) \in E$ and $(x_1 \notin K$ or $x_2 \notin K)\Big) \Rightarrow d(x_1, x_2) < \varepsilon$.

$(X,d) = (\mathbb{R}, d_{std})$. $\mathbb{R} \cong (0,1)$

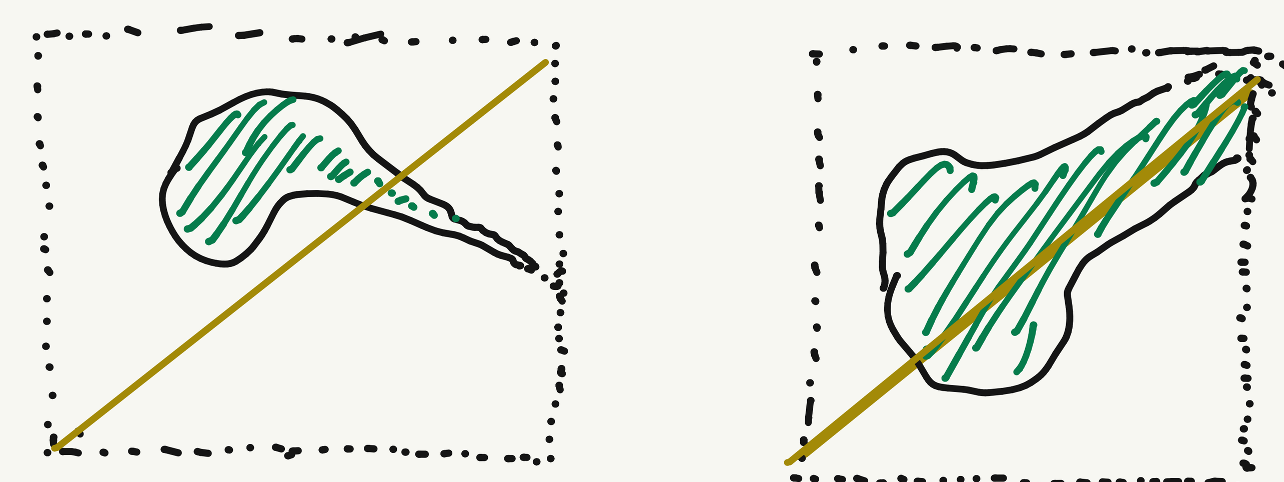


Controlled $E \subseteq X^2$

$\Delta = \{(x,x) \mid x \in \mathbb{R}\}$



Not controlled



- Topological coarse structures associated to compactification.

Let $X \in \mathrm{Top}$ paracompact, loc. cpt. Hausdorff

($(X,d)$ proper metric spaces satisfy this).

A compactification is:

$$i : X \hookrightarrow \overline{X}$$

$i$: top. emb.; $\overline{X}$: cpt. Hausdorff

$i(X)$ open dense in $\overline{X}$.

$X \subseteq \overline{X}$. The corona of the compactification is

$\partial X = \overline{X} \setminus X$ (points at infinity)

$\mathcal{E}_i = \{$ "continuously controlled sets by $i$" $\} =$ 

$\Rightarrow \mathcal{E} \subseteq \mathcal{P}(X \times X)$

$\mathcal{E} = \mathcal{P}(X \times X)$

$X$ bdd $X \sim *$

$$= \{ E \subseteq X^2 \mid \boxed{\overline{E} \setminus (X \times X)} \subseteq \Delta_{\partial X} \} =$$

in $\overline{X} \times \overline{X}$

$$= \{ E \subseteq X^2 \mid \overline{E} \cap \boxed{(\partial X \times X \cup X \times \partial X)} \subseteq \Delta_{\partial X} \} =$$

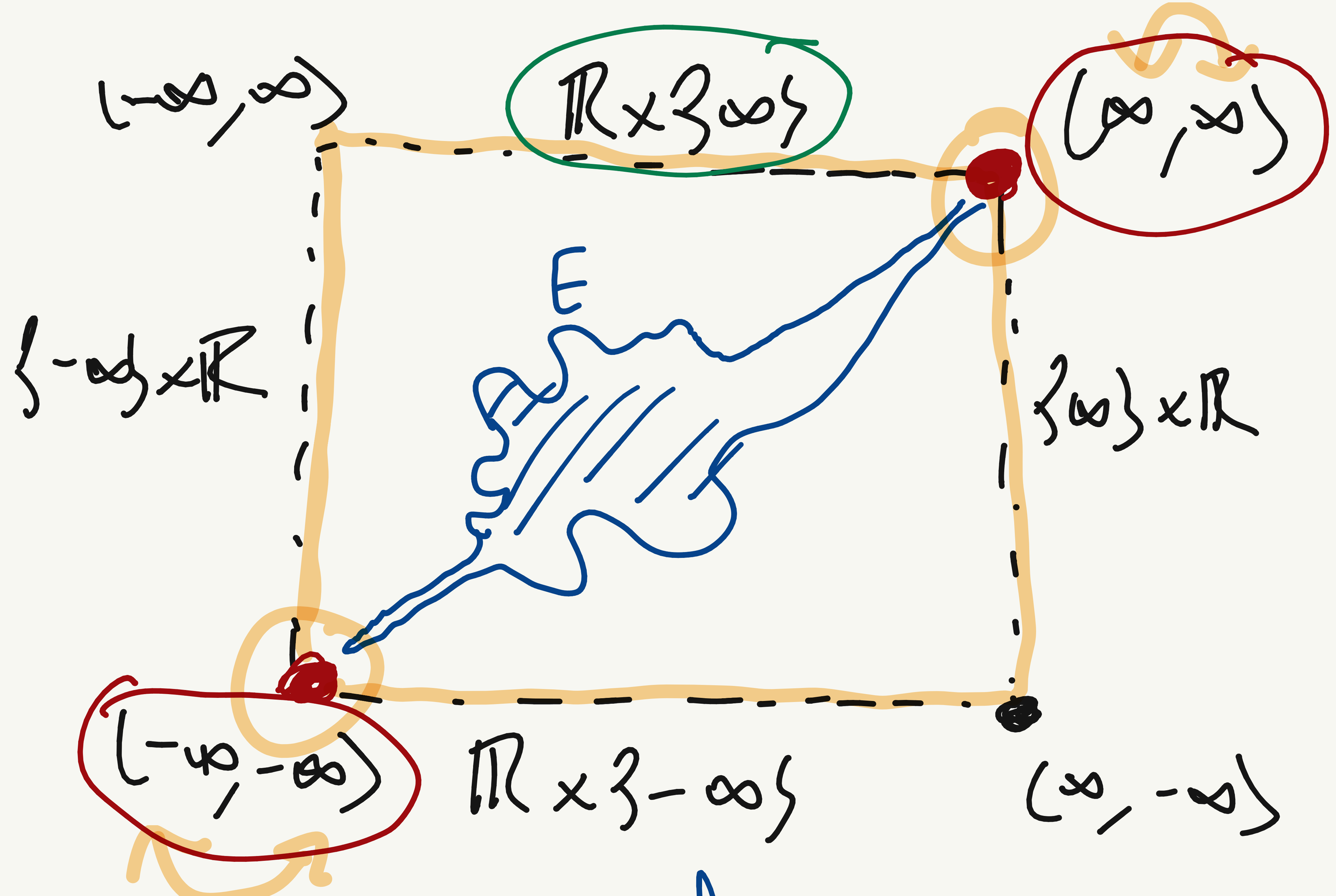
$$= \left\{ E \subseteq X^2 \;\middle|\; \begin{pmatrix} (w_1, w_2) \in \overline{E} \text{ and either} \\ w_1 \in \partial X \text{ or } w_2 \in \partial X \end{pmatrix} \Longrightarrow w_1 = w_2 \right\}$$

Example: $\boxed{X = \mathbb{R}}$, two compactifications:

$\rightarrow$ $i: \mathbb{R} \hookrightarrow \overline{\mathbb{R}} = [-\infty, \infty]$ identified w/ $(0,1) \hookrightarrow [0,1]$

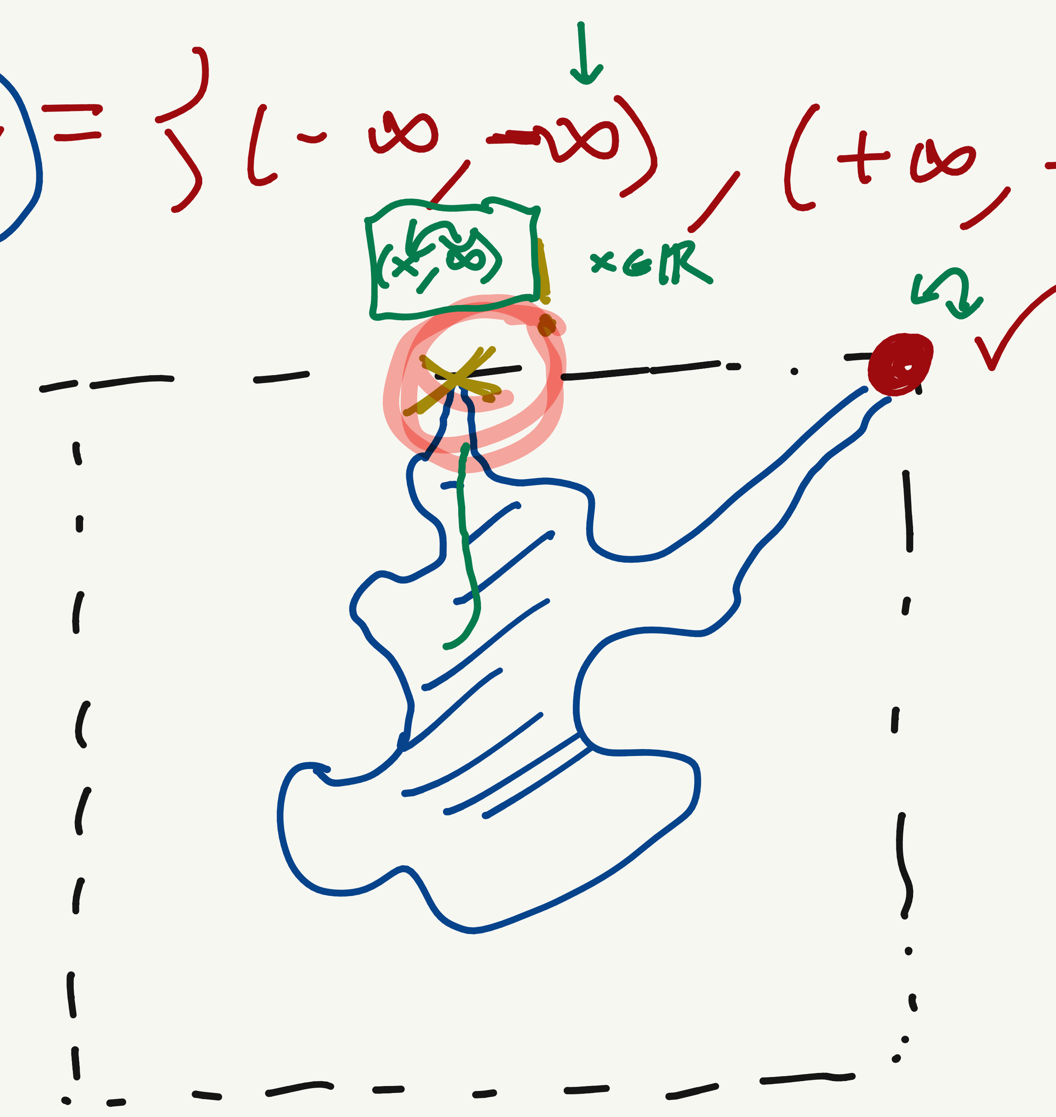
$E \subseteq \mathbb{R}^2 \subseteq \overline{\mathbb{R}}^2$

$\Delta_{\partial X} = \{ (-\infty, -\infty), (+\infty, +\infty) \}$




Controlled

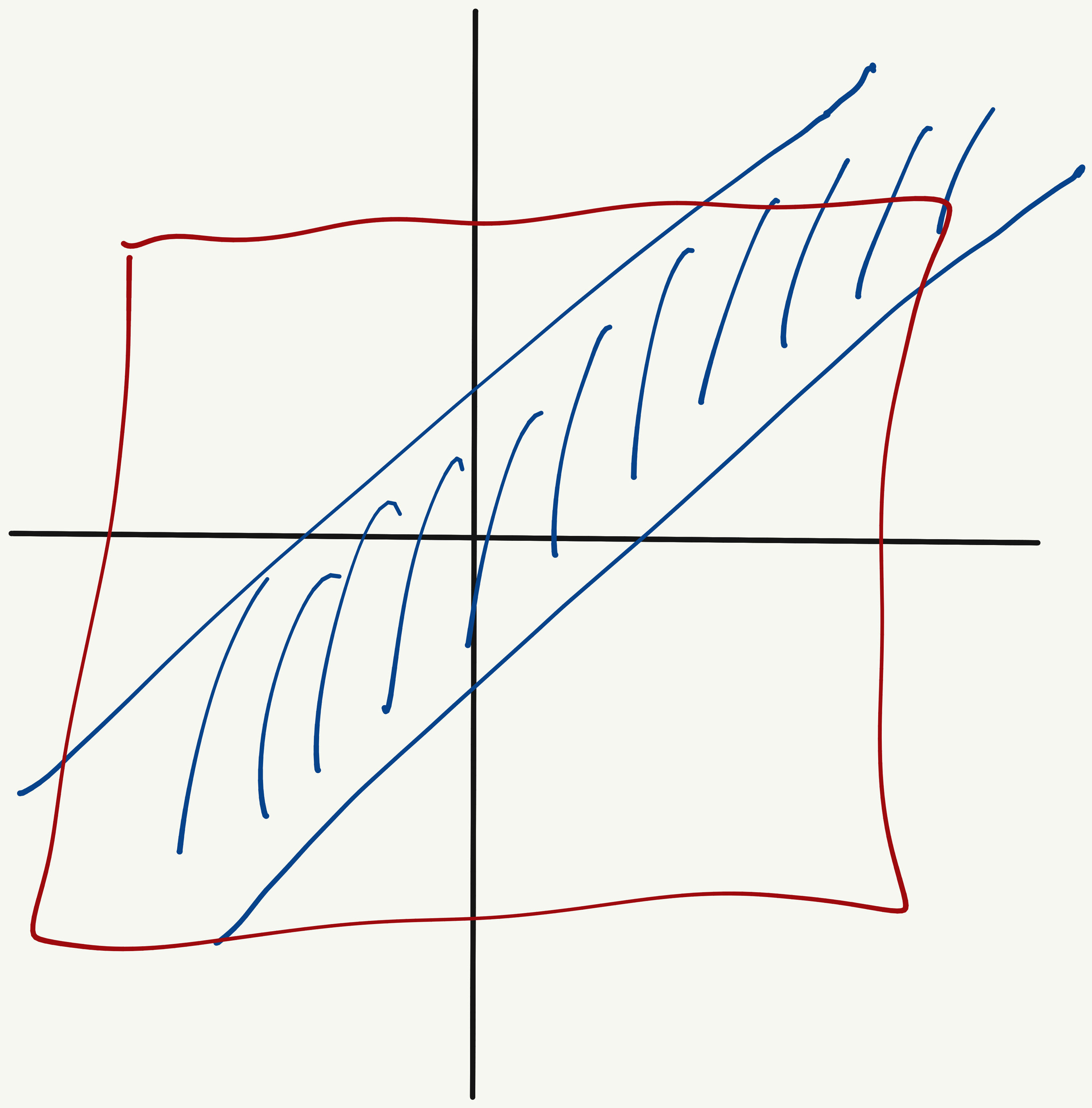
$d: \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to [0, \infty]$,




Not controlled

John Roe "Lecture on Coarse Geometry" Ch. 2.

$\mathbb{R} \cong (0,1)$ $\qquad$ $E \subseteq \mathbb{R}^2$



$G$ finite, $\quad G \curvearrowright \{*\}$ $\qquad$ $G \underset{QI}{\sim} *$ $\qquad$ (Schwarz–Milnor)