III Coarse Cohomology

3.0. Reminder

- A coarse structure on a set X is a family of binary relations $\mathcal{E} \subseteq \mathcal{P}(X \times X)$, called entourages or controlled sets, which satisfies:

  i) $E_1 \subseteq E_2 \in \mathcal{E} \Rightarrow E_1 \in \mathcal{E}$.
  ii) $E_1, E_2 \in \mathcal{E} \Rightarrow E_1 \cup E_2 \in \mathcal{E}$ } Ideal structure
  iii) $\Delta \in \mathcal{E}$.
  iv) $E_1, E_2 \in \mathcal{E} \Rightarrow E_2 \circ E_1 \in \mathcal{E}$.
  v) $E \in \mathcal{E} \Rightarrow E^{-1} \in \mathcal{E}$ } Groupoid structure.

  $X \times X$
  $(y,z)(x,y) = (x,z)$

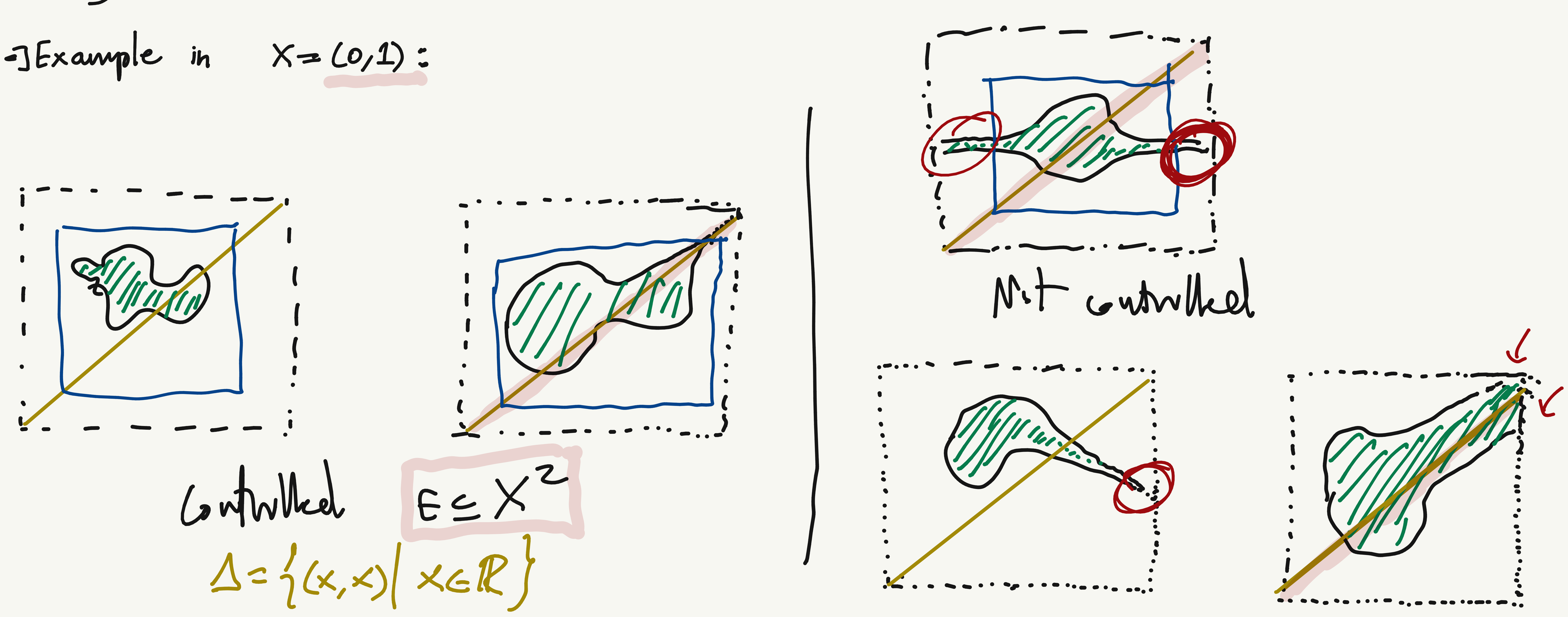
Then $(X, \mathcal{E})$ is a coarse space.

- Examples:

  i) (X,d) metric space $\rightsquigarrow$ $\mathcal{E}_d = \left\{ E \subseteq X \times X \,\middle|\, \sup\limits_{(x_1,x_2) \in E} d(x_1,x_2) < \infty \right\}$
  (Bdd) metric coarse structure

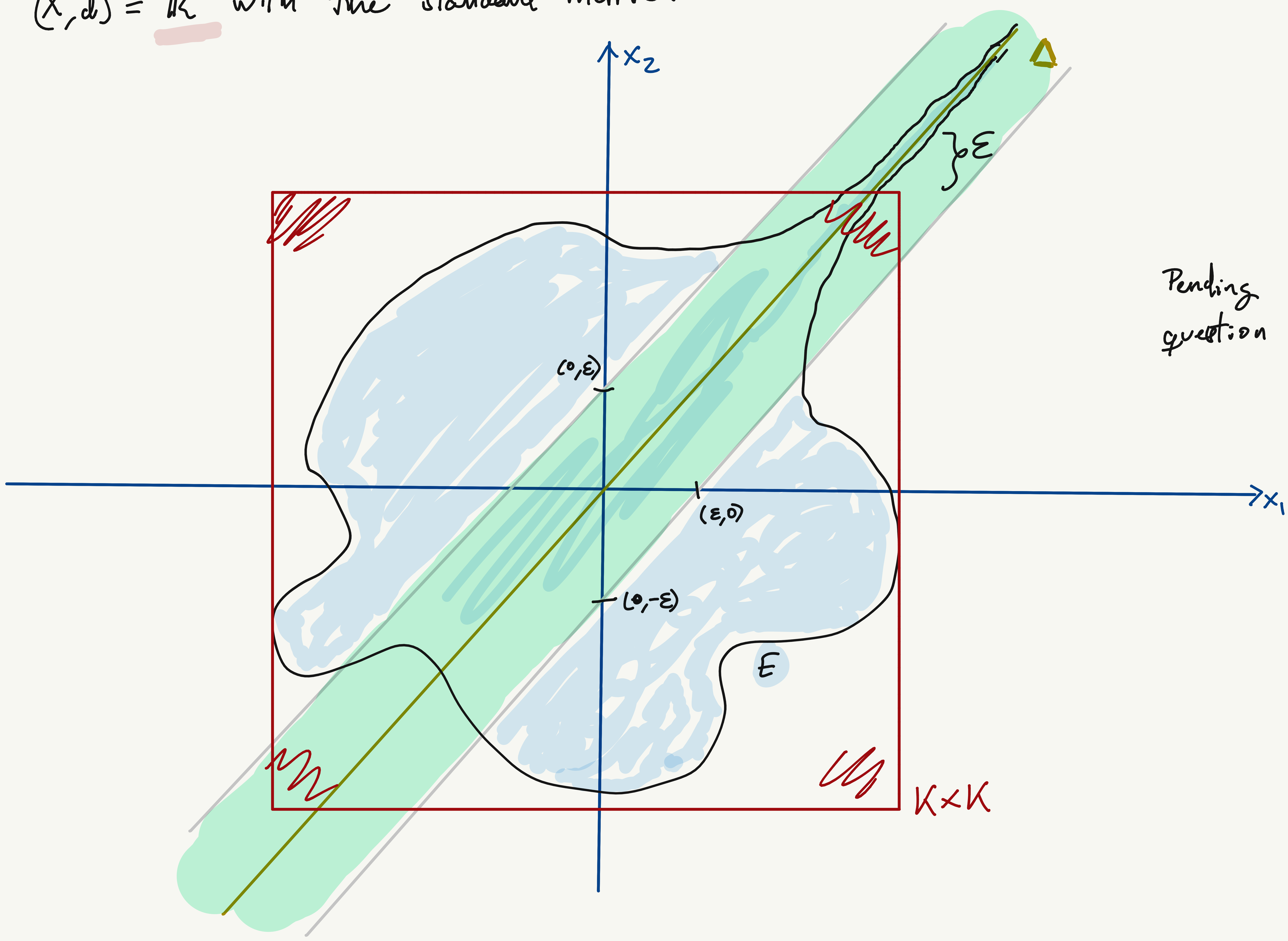
  ii) (X,d) metric space:
  $\rightsquigarrow$ $\mathcal{E}_{C_0} = \left\{ E \subseteq X \times X \,\middle|\, \forall \varepsilon > 0 \; \exists K \subseteq X \text{ cpt s.t. } d|_{E - K \times K} < \varepsilon \right\}$
  C_0 coarse structure

  - Example in $X = (0,1)$:



Here $(0,1)^2$ is metrically bdd, but not every $E \subseteq X \times X$ is C_0-controlled.

$\rightarrow$ $(X,d) = \mathbb{R}$ with the standard metric.



Pending question

$T_\varepsilon = \{(x_1,x_2)\in\mathbb{R}^2 \mid d(x_1,x_2)<\varepsilon\} = \{x_1-\varepsilon < x_2 < x_1+\varepsilon\}$

E is controlled but not bded.

iii) $i: X \hookrightarrow \overline{X}$ compactification of a paracompact locally compact Hausdorff space.

$\partial X = \overline{X} \setminus X$ "corona" of the compactification.

$\mathcal{E}_i = \{E \subseteq X\times X \mid \overline{E} \setminus X\times X \subseteq \Delta \partial X\} =$

$= \{E \subseteq X\times X \mid \overline{E} \cap (\partial X \times \overline{X} \cup \overline{X} \times \partial X) \subseteq \Delta \partial X\}$

3.1. Coarse connectedness

- Let $(X, \mathcal{E})$ be a coarse space. For $x, y \in X$,

$$x \sim y \overset{\text{def}}{\iff} \boxed{\exists\, E \in \mathcal{E} \text{ s.t. } (x,y) \in E}$$

"x and y are in the same coarsely connected component"

$F = \{(x,y)\} \subseteq X \times X$;
$F \subseteq E \in \mathcal{E}$
$\Rightarrow F \in \mathcal{E}$

$\mathcal{E} := \{(x,y)\}$

$$\iff \{(x,y)\} \in \mathcal{E}$$

$\sim$ is an equivalence relation.

The coarsely connected components of $(X, \mathcal{E})$ are the equivalence classes under $\sim$.
$(X, \mathcal{E})$ is "coarsely connected" if there is at most one coarsely conn. component.

- Example:
  - An extended metric space $(X, d: X \times X \to [0, +\infty])$, endowed w/ the bded metric coarse str., satisfies:

$\overline{\mathbb{R}} = [-\infty, +\infty]$
$d(-\infty, x) = \infty \quad \forall x \neq -\infty$
$d(+\infty, x) = \infty \quad \forall x \neq +\infty$.

$x \sim y \iff (x,y) < \infty$.
$x \sim y \iff \{(x,y)\} \in \mathcal{E}$
(X,d) is coarsely conn. iff $d(x,y) < \infty \ \forall x, y \in X$, i.e. (X,d) is a metric space.
$(\overline{\mathbb{R}}, d: \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to [0,\infty])$ has three coarsely connected components:

$$\{-\infty\}, \ \mathbb{R}, \ \{+\infty\}.$$

  - To a graph $\Gamma = (V, E)$ it corresponds an extended metric space (V, d), where

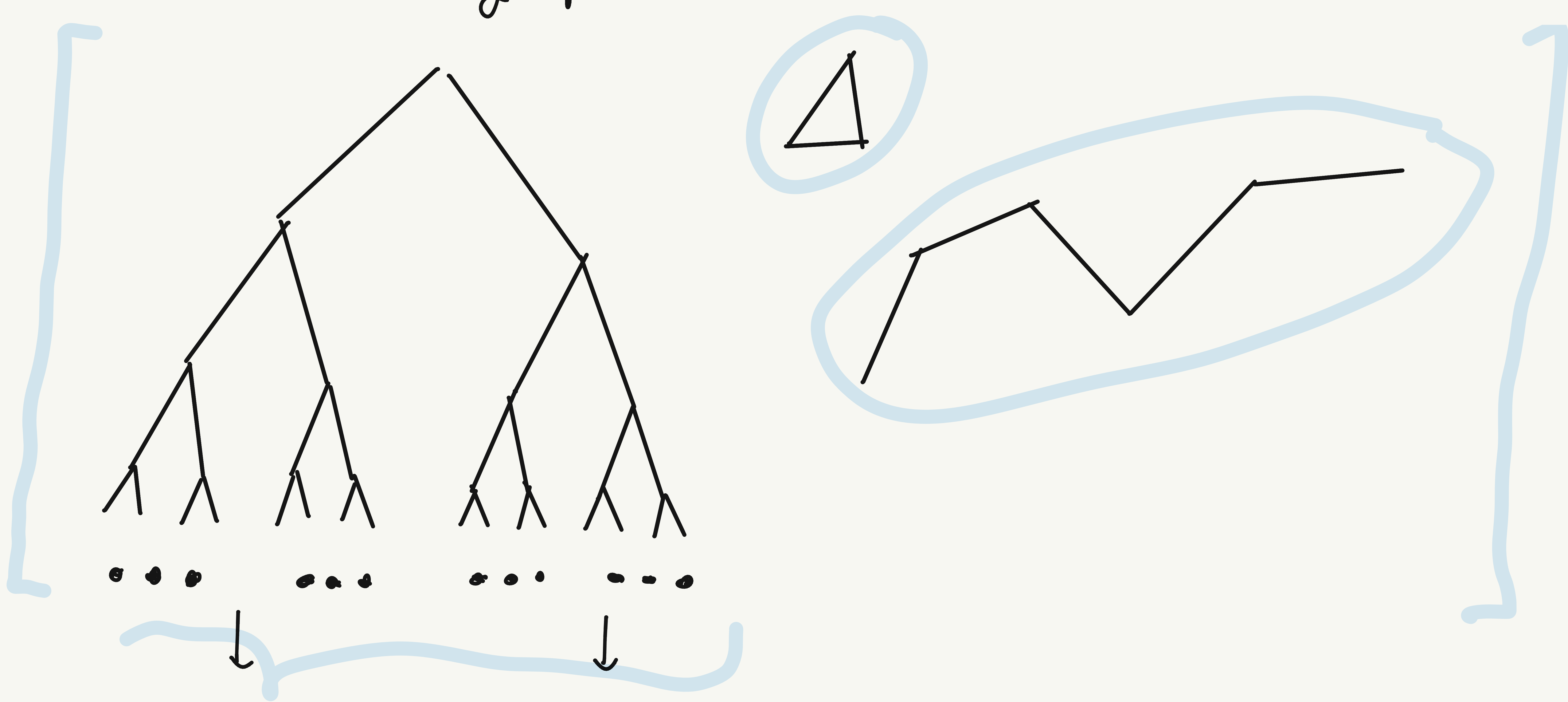
$$d(x,y) = \inf\{\operatorname{len}(\gamma) \mid \gamma \text{ path from } x \text{ to } y\} \text{ in } [0, \infty]$$

If x, y are not in the same conn. component of Γ, then

$$d(x,y) = \inf \emptyset = +\infty.$$

(V, d) is coarsely connected $\iff$ d being a metric
$\iff \Gamma = (V, E)$ is connected.

E.g.
Consider the graph $\Gamma = (V, E)$



Γ has three coarsely conn. components, one is unbdd and two bdd.

- One restrict attention to coarsely connected spaces and then argue independently on each component for non coarsely connected spaces.

3.2. closeness of maps and bounded sets

- **Def**: Let X be any set and $(Y, \mathcal{E}_Y)$ a coarse space. Two maps $f, g: X \rightrightarrows Y$ are "close", $f \approx g$, iff $\exists E \in \mathcal{E}_Y$ s.t. $(f(x), g(x)) \in E \ \forall x \in X$.

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \ ; \quad f \times g : X \times X \to Y \times Y$$
$$(x_1, x_2) \mapsto (f(x_1), g(x_2))$$

Equivalently,

$$f \approx g \iff (f \times g)(\Delta_X) = \{ (f(x), g(x)) \mid x \in X \} \text{ is controlled } (\in \mathcal{E}_Y)$$

- **Example**: Assume (Y, d) is a metric space and endow it with the bded metric coarse structure

$$\mathcal{E}_d = \{ E \subseteq Y \times Y \mid \exists r > 0 \text{ s.t. } d(x_1, x_2) \leq r \ \forall (x_1, x_2) \in E \}$$

$$X \underset{g}{\overset{f}{\rightrightarrows}} (Y, d) \ ; \quad f \approx g \iff \exists r > 0 \text{ s.t. } d(f(x), g(x)) \leq r \ \forall x \in X.$$

- $\approx$ closeness of maps is an equivalence relation on the set of maps $X \longrightarrow Y$.

- **Remark**: If (X, d) is a metric space and $\mathcal{E}_d$ its bded metric coarse structure, then, for $B \subseteq X$,

$$B \text{ is bded} \iff \operatorname{diam} B < \infty \ ; \quad \iff B \times B \in \mathcal{E}_d.$$

$$\operatorname{diam} B = \sup_{(x_1, x_2) \in B \times B} d(x_1, x_2) < \infty$$

- Def: Let $(X, \mathcal{E})$ be a coarse space. $B \subseteq X$ is said to be "bounded" if it satisfies one of the following equivalent conditions:

  i) $B \times B \in \mathcal{E}$ ($B \times B$ is controlled)

  ii) $\exists x \in X$ s.t. $B \times \{x\}$ is controlled $\left(\begin{array}{l} \text{For } \mathcal{E}_d \text{ if } (X,d) \text{ is a metric sp.,} \\ \exists r > 0 \text{ s.t. } d(y,x) \leq r \ \forall y \in B; \\ \qquad B \subseteq \overline{B}(x,r) \end{array} \right)$

  iii) $\exists E \in \mathcal{E}$ and $x \in X$ s.t. $B = E[\{x\}] = \{ y \in X \mid (x,y) \in E \}$

  iv) $i : B \hookrightarrow X$ is close to a constant map.

- Examples:

  i) (X,d) bdd metric structure $\mathcal{E}_d$.

  $B \subseteq X$ bdd wrt. $\mathcal{E}_d \iff B$ is d-bounded

  ii) (X,d) metric space. $\rightarrow \mathcal{E}_{C_0}$ coarse structure;

  $\mathcal{E}_{C_0} = \{ E \subseteq X \times X \mid \forall \varepsilon > 0 \ \exists K \subseteq X \text{ cpt s.t. } d|_{E \setminus K \times K} < \varepsilon \}$

  Let $B \subseteq X$.

  $\left[B \text{ rel. cpt} \overset{(a)}{\Longrightarrow} B \text{ is bdd wrt. } \mathcal{E}_{C_0} \overset{(b)}{\Longrightarrow} d\text{-bounded} \right]$

  (a) $\forall \varepsilon > 0$, choose $K = \overline{B}$; $\underbrace{B \times B}_{} \setminus K \times K = \emptyset$

  $B \times B$ controlled $\rightarrow B$ bdd wrt. $\mathcal{E}_{C_0}$.

  (b) Choose $\varepsilon > 0$. Since $B \times B \in \mathcal{E}_{C_0}$, $\exists K \subseteq X$ cpt. s.t.

  $d|_{B \times B \setminus K \times K} < \varepsilon$.

  $B \times B \setminus K \times K = (B \setminus K) \times B \cup B \times (B \setminus K)$

  If $B \setminus K = \emptyset$, $B \subseteq K$ and K is cpt, then B is bdd.

  otherwise, choose $x \in B \setminus K$, and then for $y \in B$,

  $(x,y) \in (B \setminus K) \times B$, $d(x,y) < \varepsilon$; $B \subseteq B(x, \varepsilon)$, B d-bdd

  Conclusion: Iff (X,d) is a proper metric space,

  B bdd wrt. $\mathcal{E}_{C_0} \iff B$ d-bdd $\iff B$ rel. cpt.

iii) $i : X \hookrightarrow \overline{X}$ compactification. $\partial X = \overline{X} \setminus X$
"corona"

$$\mathcal{E}_i = \{ E \subseteq X \times X \mid \overline{E} \setminus X \times X \subseteq \Delta_{\partial X} \}$$

$\overline{X} \times \overline{X}$

Let $B \subseteq X$. Then

$$B \text{ bdd} \iff B \times B \in \mathcal{E}_i \iff \underbrace{(\overline{B \times B}) \cap (\partial X \times \overline{X} \cup \overline{X} \times \partial X)}_{(\overline{B} \cap \partial X) \times \overline{B} \,\cup\, (\overline{B} \times \overline{B} \cap \partial X)} \subseteq \Delta_{\partial X}$$

$$(\overline{B} \cap \partial X) \times \overline{B} \cup (\overline{B} \times \overline{B} \cap \partial X) \subseteq \Delta_{\partial X}$$

If $\overline{B} \cap \partial X = \phi$, then the condition is satisfied.
so B is bdd.

If $\overline{B} \cap \partial X \neq \phi$, in particular $\overline{B} \neq \phi$, and so $B \neq \phi$.
We choose $w \in \overline{B} \cap \partial X$ and $y \in B$. Then

$$(w, y) \in (\overline{B} \cap \partial X) \times \overline{B} \setminus \Delta_{\partial X}$$

So B is not bdd.

Then:

$$B \text{ is bdd} \iff \boxed{\overline{B} \cap \partial X = \phi} \iff \overline{B}^{\overline{X}} \subseteq X$$

3.3. Maps between coarse spaces

• **Def**: Let $f : (X, \mathcal{E}_X) \longrightarrow (Y, \mathcal{E}_Y)$ be a map between coarse spaces.

·) f is "controlled" if $\forall E \in \mathcal{E}_X$, $\underbrace{(f \times f)(E)}_{\{(f(x_1), f(x_2)) \mid (x_1, x_2) \in E\}} \in \mathcal{E}_Y$.

f is "controlled" if $\forall E \in \mathcal{E}_X$ $\exists F \in \mathcal{E}_Y$ s.t.

$$(x_1, x_2) \in E \implies (f(x_1), f(x_2)) \in F.$$

·) f is "proper" if $\forall B \subseteq Y$ bdd, $f^{-1}(B) \subseteq X$ is bdd.

·) f is a "coarse map" if it is controlled and proper.

The category "Coarse" has:

·) objects: Coarse spaces $(X, \mathcal{E}_X)$

·) $\mathrm{Coarse}(X, Y) = \{ \text{coarse maps } X \to Y \}$ | Composition of coarse maps is coarse.

Coarse

• Weak equivalences (coarse equivalence)
Category w/ weak equivalences
• Fib, Cofib?

$(X, \mathcal{E})$

$(X, \mathcal{E}_X) \longrightarrow (Y, \mathcal{E}_Y)$

Top

• Weak eq.
• Fib.
• Cofib.

(X, τ)

$(X, \tau_X) \longrightarrow (Y, \tau_Y)$

$X \overset{f}{\underset{g}{\rightrightarrows}} Y$; $f \simeq g$.

Coarse/$\approx$

$f_1 \approx f_2$, $g_1 \approx g_2 \Rightarrow g_1 f_1 \approx g_2 f_2$

ob: $(X, \mathcal{E})$ coarse spaces

$\mathrm{hom}(X, Y) = \{ [f] \mid f : X \to Y \text{ coarse} \}$

eq. class up to closeness.

Top/$\simeq$

ob: (X, τ) top. spaces

$\mathrm{hom}(X, Y) = \{ [f] \mid f : X \to Y \text{ continuous} \}$

eq. class under $\simeq$

$$\begin{tikzcd} \mathrm{Coarse} \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{D} \\ \mathrm{Coarse}/\approx \arrow[ur, dashed, "\tilde{\mathcal{F}}"'] & \end{tikzcd}$$

functor

Coarse invariants

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "\mathcal{F}"] \arrow[d] & \mathcal{D} \\ \mathrm{Top}/\simeq \arrow[ur, dashed, "\bar{\mathcal{F}}"'] & \end{tikzcd}$$

Homotopy invariants

If $\mathcal{F}: \mathbf{Coarse} \to \mathcal{D}$ is a coarse invariant, then

$$X \underset{ce}{\simeq} Y \implies \mathcal{F}(X) \cong \mathcal{F}(Y).$$

(isomorphic in $\mathbf{Coarse}/\approx$)

- $f: (X, \mathcal{E}) \longrightarrow (Y, \underbrace{\mathcal{P}(Y \times Y)}_{\text{maximal coarse structure}})$

f is $\boxed{\text{controlled}}$ $\forall\, \mathcal{E}$ coarse str on X

f is proper iff $(X, \mathcal{E})$ is bdd, iff $\mathcal{E} = \mathcal{P}(X \times X)$.

Control: Ob: $(X, \mathcal{E})$ coarse spaces

hom: controlled maps.

$$\Big(f_i : \underset{\text{set}}{X} \longrightarrow (Y_i, \mathcal{E}_i)\Big)_{i \in I} \quad \mathcal{U}\text{-source};$$

$\mathcal{E}_{ini}$ coarse structure on X; the largest coarse structure on X such that f_i is controlled $\forall i \in I$

$(X,d) \to$ Bdd metric structure $\mathcal{E}_d$

i.) $\boxed{\text{Connectedness}}$

ii.) Bddness

$B \subseteq X$ is bounded wrt. $\mathcal{E}_d \iff B\times B \in \mathcal{E}_d \iff \exists r>0$ st. $d(x_1,x_2) \le r \quad \forall (x_1,x_2) \in B\times B \iff \exists r / \operatorname{diam} B \le r \iff B$ d-bdd

i) Coarse connectedness vs path-conn. vs connectedness.

(X,d) is coarsely connected iff $d(x,y) < \infty \quad \forall x,y \in X$

Path-connected $\Rightarrow$ coarsely conn. (Heine–Cantor Theorem).

$\not\Leftarrow$ E.g. G f.g. groups:

$(\mathbb{Z},d)$ is coarsely conn but it is not path-conn.

$Cay(G,S) \rightsquigarrow (G,d_S)$ not path-conn., but coarsely equivalent to the geometric realization of $Cay(G,S)$, which is path-conn.

$\Longrightarrow$ Question; Given a metric space X (which is then coarsely conn.), is there a path-conn. metric space Y st. $X \underset{c.e}{\simeq} Y$?

$X = \{2^n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ w/ the inherited metric,

X is not coarsely equivalent to a path-conn. metric space.
because X is not r-connected for any $r>0$.
(Y is "r-connected" if $\forall x,y \in Y \ \exists x_0 = x, x_1, \dots, x_n = y \in Y$ such that $d(x_i,x_{i+1}) \le r \quad \forall i = 0,\dots,n-1$)