4. Computation of Coarse Cohomology

4.0. Reminder

- $f: X \to Y$ is "controlled" if $\forall E \in \mathcal{E}_X$, $(f \times f)(E) \in \mathcal{E}_Y$.
  " "proper" if $\forall B \subseteq Y$ bdd, $f^{-1}(B)$ is bdd in X
  " "coarse" if it is controlled and proper.

  Coarse ; $\text{coarse}/_{\approx}$

  $f: X \to Y$ is a c.equivalence if $\exists g: Y \to X \in$ Coarse s.t.

  $gf \approx id_X \qquad fg \approx id_Y$.

4.1. Coarse Cohomology

The def'n given here is equivalent to the original by John Roe ; "Lectures on Coarse Geometry", §5.

- $X \in$ Coarse, G abelian group.
  $C^*(X; G)$ "basic cochain complex" (of X w/ coeff. in G)
  $C^n(X; G) = \{ \boxed{\varphi: \underbrace{X^{n+1}} \to G} \}$ $\qquad \overset{\downarrow}{D_\bullet} X \quad D_n X = X^{n+1}$
  $(x_0, \dots, x_n) \mapsto \varphi(x_0, \dots, x_n) \in G$
  n-simplices (n+1 vertices)

  $\delta: C^n(X; G) \to C^{n+1}(X; G)$;
  $\varphi: X^{n+1} \to G \mapsto \delta\varphi: X^{n+2} \to G$;
  $\underbrace{(x_0, \dots, x_{n+1})}_{\sigma} \mapsto \sum_{i=0}^{n+1} (-1)^i \varphi(\underbrace{x_0, \dots, \hat{x_i}, \dots, x_{n+1}}_{\text{i-th face of } \sigma})$

  $\delta^2 = 0$;
  $C^*(X)$ is contractible up to degree zero ;

- Def'n : A cochain $\varphi: X^{n+1} \to G$ is a "coarse cochain" if
  $\forall E \in \mathcal{E}_X$, $\underbrace{E[\Delta]} \cap \operatorname{supp}\varphi$ is bdd

  $\Delta = \{(x, \dots, x) \mid x \in X\} \subseteq X^{n+1}$;

  $E[\Delta] = \{\sigma = (x_0, \dots, x_n) \mid \exists x \in X \text{ s.t. } (x, x_i) \in E \ \forall i = 0, \dots, n\}$

  $E[\Delta] \cap \operatorname{supp}\varphi \subseteq X^{n+1}$

"$E[\Delta] \cap \operatorname{supp}\varphi$ bdd" means $\exists \tilde{x} \in X$ and $G \in \mathcal{E}_X$ s.t.

$$E[\Delta] \cap \operatorname{supp}\varphi \subseteq G[(\tilde{x}, \dots, \tilde{x})] = \{\sigma = (x_0, \dots, x_n) \mid (\tilde{x}, x_i) \in G \ \forall i = 0, \dots, n\}$$

Let

$$CX^n(X; G) = \{\varphi : X^{n+1} \to G \mid E[\Delta] \cap \operatorname{supp}\varphi \text{ is bdd } \forall E \in \mathcal{E}_X\}$$

$CX^n(X) \subseteq C^n(X)$. $\quad \delta(CX^n(X)) \subseteq CX^{n+1}(X)$

<u>Lemma</u>: $CX^*(X)$ is a subcomplex of $C^*(X)$.

Let $\varphi \in CX^n(X)$. Let $E \in \mathcal{E}_X$. We want to check that $E[\Delta] \cap \operatorname{supp}(\delta\varphi)$ is bdd. We know $E[\Delta] \cap \operatorname{supp}(\varphi)$ is bdd; $\exists \tilde{x} \in X$ and $G \in \mathcal{E}_X$ s.t. $E[\Delta] \cap \operatorname{supp}\varphi \subseteq G[(\tilde{x}, \dots, \tilde{x})]$.

Let $\sigma \in \boxed{E[\Delta]} \cap \operatorname{supp}(\delta\varphi)$; $\sigma = (x_0, \dots, x_{n+1})$: Let $x \in X$ s.t. $(x, x_j) \in E$ $\forall j = 0, \dots, n+1$

Since $(\delta\varphi)(\sigma) \neq 0$, $\exists i = 0, \dots, n+1$ s.t. $\varphi(\sigma_i) \neq 0$;

$\sigma_i = (x_0, \dots, \hat{x_i}, \dots, x_{n+1})$, since $\sigma \in E[\Delta]$, $\sigma_i \in E[\Delta]$;

$\sigma_i \in E[\Delta] \cap \operatorname{supp}\varphi \subseteq G[(\tilde{x}, \dots, \tilde{x})]$;

$\forall j = 0, \dots, \hat{i}, \dots, n+1$, $(\tilde{x}, x_j) \in G$. Let $j \neq i$,

$$\begin{tikzcd} & x_j & x \arrow[l, "E"'] \arrow[d, "E"] \\ \tilde{x} \arrow[ur, "G"] & & x_i \end{tikzcd} \qquad (\tilde{x}, x_i) \in E \circ E^{-1} \circ G$$

$(\tilde{x}, x_j) \in G \cup (E \circ E^{-1} \circ G)$ $\forall j = 0, \dots, i, \dots, n+1$.

$E[\Delta] \cap \operatorname{supp}(\delta\varphi)$ is bdd. ∎

Contractibility of $C^*(X)$

Assume $X \neq \emptyset$. Choose a basepoint $p \in X$.

$$s^n : C^n(X) \longrightarrow C^{n-1}(X) \;; \qquad n \geq 1$$

$$\varphi : X^{n+1} \to G \;\mapsto\; (s\varphi) ; (x_0, \dots, x_{n-1}) \mapsto \varphi(p, x_0, \dots, x_{n-1})$$

$$\begin{tikzcd}
 & C^n(X) \arrow[r, "\delta^n"] \arrow[dl, "s^n"'] \arrow[d, phantom, "(+)"] & C^{n+1}(X) \arrow[dl, "s^{n+1}"] \\
C^{n-1}(X) \arrow[r, "\delta^{n-1}"] & C^n(X) &
\end{tikzcd}$$

$$s^{n+1}\delta^n + \delta^{n-1} s^n = id_{C^n(X)} \qquad \boxed{\text{for } n \geq 1.}$$

$$0 \to G \xrightarrow{\varepsilon} C^0(X) \to C^1(X) \to C^2(X) \to \cdots \quad \boxed{\tilde{C}^*(X)}$$

$$\boxed{\tilde{C}^*(X)} \to \tilde{H}^*(X)$$

- Def'n: $HX^*(X)$ "coarse cohomology of X w/ coeff. in G" is defined as the cohomology of $CX^*(X;G)$.

- Prop: $HX^* : \text{Coarse}^{op} \longrightarrow Ab$

$$f : X \to Y \;\rightsquigarrow\; f^* ; HX^*(Y) \longrightarrow HX^*(X)$$

Moreover, HX^* is a "coarse invariant":

$$[\, f, g : X \rightrightarrows Y \;;\; f \approx g \,] \Rightarrow f^* = g^*$$

Therefore, if f is a coarse equivalence, f^* is an isomorphism.

$$\begin{tikzcd}
\text{Coarse}^{op} \arrow[r, "HX^*"] \arrow[d] & Ab \\
\text{Coarse}^{op}/\approx \arrow[ur, dashed] &
\end{tikzcd}$$

- $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in Coarse.

$f \approx g \overset{\text{def}}{\Longleftrightarrow} (f \times g)(\Delta) \in \mathcal{E}_Y$

"f and g are close"

$\Longleftrightarrow \exists F \in \mathcal{E}_Y$ s.t. $(f(x), g(x)) \in F \quad \forall x \in X$.

$d_\infty(f, g) < \infty$.

- How to compute?

4.2: Topological bornological spaces.

- Def'n: A bornology on a set X is a family $\mathcal{B}$ of subsets of X s.t.:

i) $X = \bigcup_{B \in \mathcal{B}} B \qquad x \in \{x\}$

ii) $B_1, B_2 \in \mathcal{B} \implies B_1 \cup B_2 \in \mathcal{B}$

iii) $B_1 \subseteq B_2 \in \mathcal{B} \implies B_1 \in \mathcal{B}$

Elements of $\mathcal{B}$ are called "bounded sets".

- Example: (X, d); $\mathcal{B} = \{$bdd sets$\}$ is a bornology.
- Example: (X, τ) top. space. Assume it is loc. cpt. Hausdorff. Then $\mathcal{B} = \{$rel. cpt. sets$\}$ is a bornology.

- Def'n: Let (X, τ) be a top. space. A bornology $\mathcal{B}$ on X is "compatible w/ τ" if:

i) $B \in \mathcal{B} \implies \overline{B} \in \mathcal{B}$.

ii) $B \in \mathcal{B} \implies \exists N \in \mathcal{B} \cap \tau$ s.t. $B \subseteq N$.

$(X, \tau, \mathcal{B})$ is a topological bornological space.

- Rmk: If $(X, \tau, \mathcal{B})$ is top. born. space, then every rel. cpt set is bounded:

If B is rel. cpt, $\overline{B}$ is compact.

$\forall x \in \overline{B}$, $\exists N_x$ open bded st. $x \in N_x$: $\{N_x \mid x \in \overline{B}\}$ open cover $\overline{B}$. let $\{N_{x_1}, \dots, N_{x_K}\}$ finite subcover.

$$\overline{B} \subseteq \bigcup_{i=1}^{K} N_{x_i}$$

$$\mathrm{Top}^{\mathrm{lc.\,Haus}} \hookrightarrow \mathrm{TopBorn} \qquad \text{proper homotopy theory.}$$

- TopBorn is the category where:

  Ob: $(X, \tau, \mathcal{B})$ top. born. spaces.

  Morphisms: $f : X \to Y$ is a morphism if it continuous and $\underbrace{\text{proper}}$

  $\forall B \in \mathcal{B}_Y$, $f^{-1}(B) \in \mathcal{B}_X$.

- Bornological cohomology.

$$C\mathcal{B}^*(X) \hookrightarrow C^*(X)$$

$$C\mathcal{B}^n(X) = \left\{ \varphi : X^{n+1} \to G \;\middle|\; \Delta \cap \overline{\operatorname{supp} \varphi} \text{ is bded} \right\}$$

bornological n-cochains

$C\mathcal{B}^*(X)$ is a cochain subcomplex of $C^*(X)$.

The bornological cohomology of X is the cohomology of $C\mathcal{B}^*(X)$;

$$H\mathcal{B}^*(X) = \frac{\operatorname{Ker}\left(\delta|_{C\mathcal{B}^*(X)}\right)}{\operatorname{Im}\left(\delta|_{C\mathcal{B}^{*-1}(X)}\right)}$$

• **Prop**: $HB^*: \mathrm{TopBorn}^{op} \longrightarrow \mathrm{Ab}$

$$f: X \to Y \rightsquigarrow HB^*(f): HB^*(Y) \longrightarrow HB^*(X).$$

$\boxed{HX^*}$ related to $\boxed{HB^*}$?

How ?

$HX^*: \mathrm{Coarse}^{op} \longrightarrow \mathrm{Ab}$

Mayer–Vietoris seq. in the literature

$HB^*: \mathrm{TopBorn}^{op} \longrightarrow \mathrm{Ab}$

Mayer–Vietoris ?

$$\begin{tikzcd}
& X = A \cup B & \\
A \arrow[ur] & & B \arrow[ul] \\
& A \cap B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$\boxed{X = \mathring{A} \cup \mathring{B}}$

$A \cup B = X \quad \overset{}{\underset{\text{relax}}{\rightsquigarrow}} \quad A \cup B$ is "coarsely dense" on X

$$\exists E \in \mathcal{E}_X \text{ s.t. } \underbrace{E[A \cup B]}_{\{y \in X \mid \exists x \in A \cup B \text{ for which } (x,y) \in E\}} = X$$

- "Coarse Cohomology and Index Theory of Complete Riem. mnflds" Chapters 2, 3
- "Lectures on Coarse Geometry" Chapter 5.

John Roe

$$HX^*(M) \longrightarrow H_c^*(M) \longrightarrow H^*(M)$$