V. From Bornological to Coarse Cohomology

0. Reminder

- TopBorn: $\left[\begin{array}{ll} (X, \tau, \mathcal{B}) & \text{topological bornological space} \\ f: X \to Y & \text{cont. proper} \end{array}\right.$

- $CB^*: \text{TopBorn}^{op} \longrightarrow \text{CoCh}$

$$X \longmapsto CB^*(X;G)$$

$$CB^n(X;G) = \{\varphi: X^{n+1} \to G \mid \Delta \cap \operatorname{supp}\varphi \ \text{bdd}\}.$$

$$HB^*: \text{TopBorn}^{op} \longrightarrow \text{CoCh}$$

- Thm (R. 2025): For any $X \in \text{TopBorn}$,

$$HB^*(X;G) \cong \operatorname*{colim}_{B \in \mathcal{B}} \overline{H}^{*-1}(X \setminus B; G)$$

- Examples: $\mathbb{N} \cong \mathbb{Z}$ in TopBorn;

$$HB^*(X;G) = 0 \quad \text{for } * \neq 1$$

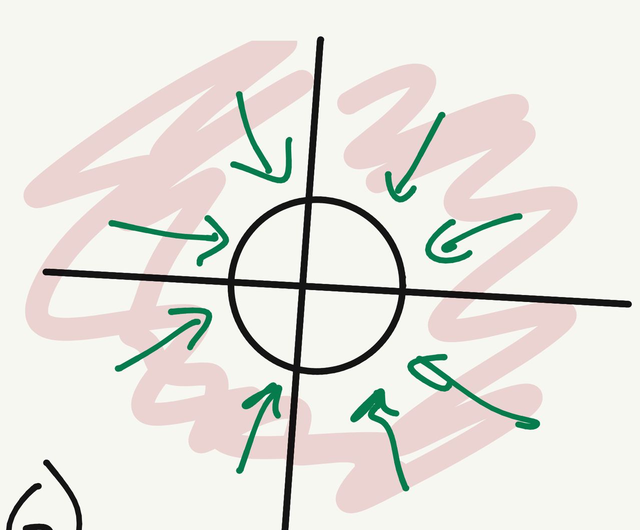
Examples: $X = [0,\infty)$.

For each $n \in \mathbb{N}$, $B_n = [0,n]$; $(B_n)_{n\in\mathbb{N}}$

$$HB^*(X;G) \cong \operatorname*{colim}_{n\to\infty} \overline{H}^{*-1}((n,\infty), G) \cong$$

$$\cong \overline{H}^{*-1}(\{*\}; G) = 0$$

•) $X = \mathbb{R}^n$; $n \geq 1$;



$$B_n = B(0;n);$$

$$HB^*(X;G) \cong \operatorname{colim} \overline{H}^{*-1}(\underbrace{\mathbb{R}^n \setminus B(0,n)}_{S^{n-1}}; G)$$

$$\cong \overline{H}^{*-1}(S^{n-1}; G) = \begin{cases} G; & * = n \\ 0; & \text{otherwise} \end{cases}$$

$$\mathbb{R}^n \cong \mathbb{R}^m \Rightarrow HB^*(\mathbb{R}^n) \cong HB^*(\mathbb{R}^m) \Rightarrow n = m.$$

Coarse vs bornological cohomology

- Assume (X,d) is a metric space.

  ↳ Coarse space w/ metric str. $\rightsquigarrow HX^*(X)$

  ↳ top. born. space $\rightsquigarrow HB^*(X)$.

$$\begin{tikzcd}
 & \rightsquigarrow HB^*(X) & & \\
C_0^*(X) \arrow[r, hook] \arrow[rr, bend left, red] \arrow[drrr, bend right, blue, "\wr", "*-1"'] & CB^*(X) \arrow[r, hook] & C^*(X) \arrow[d, red, two heads] \arrow[r, squiggly] & \begin{cases} G; & *=0 \\ 0; & *\geq 1 \end{cases} \\
 & CX^*(X) \arrow[u, hook] \arrow[d, squiggly] & C^*_{cs}(X) \arrow[r, squiggly] & H^*(X) \\
 & HX^*(X) & &
\end{tikzcd}$$

Since $CX^*(X) \hookrightarrow CB^*(X)$; there is an induced map $HX^*(X) \xrightarrow{\Sigma} HB^*(X)$. When is it an iso?

- $\mathbb{N} \sim [0,\infty)^{\psi\ \mathbb{Z}}$ c.e. in Coarse

  $\mathbb{N} \nsim [0,\infty)$ in TopBorn
  $\begin{array}{c} \rotatebox{90}{$\cong$} \\ \mathbb{Z} \end{array}$

- <u>Def'n</u>: X is "uniformly contractible at infinity" if $\exists \rho, \mu : [0,\infty) \to [0,\infty)$ and a basepoint p s.t.:

$$[B \subseteq X,\ \operatorname{diam} B = r \text{ and } d(p,B) \geq \mu(r)] \Rightarrow \begin{bmatrix} B \text{ is contractible inside a set of} \\ \text{diameter } \rho(r) \end{bmatrix}$$

$(\exists B';\ B \hookrightarrow B')$.

* **Thm** (Banerjee, 2024): If X is "uniformly contractible at infinity", then

$$\xi : HX^*(X) \xrightarrow{\cong} \underbrace{HB^*(X)}$$ is an isomorphism.

Consequence: $$\boxed{HX^*(X) \cong \operatorname*{colim}_{B \in \mathcal{B}} \overline{H}^{*-1}(X \setminus B)}$$

If $p \in X$ is a basepoint, then

$$HX^*(X) \cong \operatorname*{colim}_{r \to \infty} \overline{H}^{*-1}(X \setminus B(p,r))$$

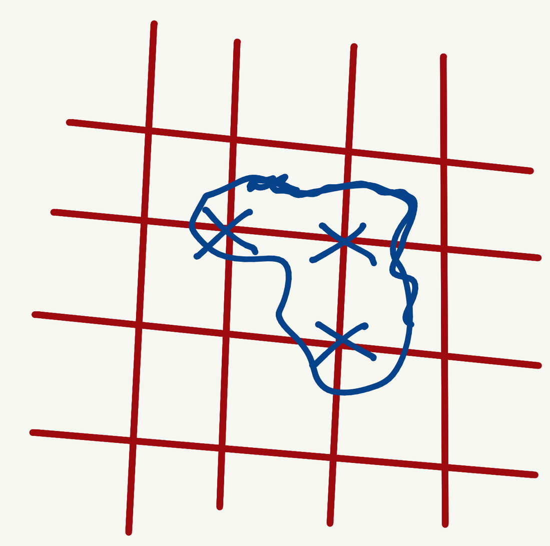
* Examples:

-) $\mathbb{R}^n$ is unif. contr. at infinity;

$$HX^*(\mathbb{R}^n) \cong HB^*(\mathbb{R}^n) \cong \begin{cases} G ; & * = n \\ 0 ; & \text{otherwise} \end{cases}$$

-) $\mathbb{Z}^n$ is not unif. contr. at infinity;

$\mathbb{Z}^n \underset{c.e.}{\sim} \mathbb{R}^n$;



$$HX^*(\mathbb{Z}^n) \cong \begin{cases} G ; & * = n \\ 0 ; & \text{otherwise} \end{cases}$$

Consequence: $\mathbb{Z}^n \underset{c.e.}{\sim} \mathbb{Z}^m \implies n = m$

classification of abelian f.g. groups:

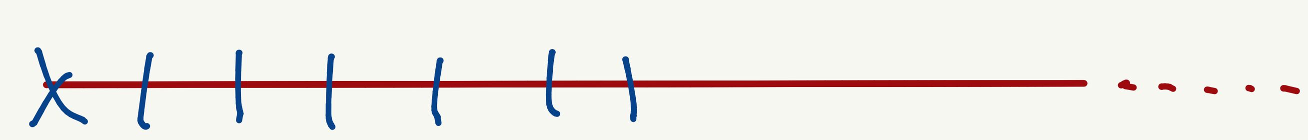
If A is abelian f.g., then $A = \mathbb{Z}^r \oplus T$, where T is a torsion group. T is finite; $A \sim \mathbb{Z}^r$;

Consequence: If A_1, A_2 abelian f.g. groups, then

$$A_1 \underset{c.e.}{\sim} A_2 \iff rk(A_1) = rk(A_2).$$

$$\begin{array}{ccc} A_1 & \underset{c.e.}{\sim} & A_2 \\ \wr & & \wr \\ \mathbb{Z}^n & \sim & \mathbb{Z}^m \end{array} \implies n = m$$

-) $\mathbb{N} \underset{c.e.}{\sim} [0,\infty)$;



$$HX^*(\mathbb{N}) \cong HX^*([0,\infty)) \overset{\text{thm}}{\cong}$$

unif. contractible at infinity

$$HX^*(\mathbb{N}) \cong HX^*([0,\infty)) = \operatorname*{colim}_{n\to\infty} \overline{H}^{*-1}(X \setminus B(0,n)) = \overline{H}^{*-1}(*) = 0 \quad , * \in \mathbb{Z}$$

Mayer-Victoris sequence

$HX_*: \mathrm{Coarse} \longrightarrow \mathrm{Gr}_{\mathbb{Z}}\mathrm{Ab}$;

(homology)

$$\begin{tikzcd} & X = A \cup B & \\ A \arrow[ur] & & B \arrow[ul] \\ & A \cap B \arrow[ul] \arrow[ur] & \end{tikzcd}$$

[Paul D. Mitchener "Coarse homology theories"

Eilenberg–Steenrod axioms for coarse spaces $\downarrow$ homology theory of.]

$(h_n : \mathrm{Top} \to \mathrm{Ab})_{n\in\mathbb{Z}}$; $\updownarrow$

$(hX_n : \mathrm{Coarse} \to \mathrm{Ab})_{n\in\mathbb{Z}}$

$A, B \subseteq X$; (A,B) is excisive if $\forall E \in \mathcal{E}$, $\exists F \in \mathcal{E}$ s.t.

$$\boxed{\underbrace{E[A] \cap E[B]}} \subseteq \boxed{F[A \cap B]}$$

Products in Coarse Cohomology

$(X, \mathcal{E})$ coarse space ; R a coefficient ring ;

$$HX^*(X) = HX^*(X;R)$$

$$CX^*(X;R) \hookrightarrow \boxed{(C^*(X;R), \smile)}$$

$$\smile : C^n(X) \otimes C^m(X) \longrightarrow C^{n+m}(X)$$

$$(\varphi, \psi) \mapsto (\varphi \smile \psi)(x_0, \dots, x_{n+m}) =$$

$$= \varphi(x_0, \dots, x_n)\, \psi(x_n, \dots, x_{n+m})$$

$\varphi \in CX^n$, $\psi \in CX^m \implies \varphi \smile \psi \in CX^{n+m}$

$(CX^*(X), \smile) \rightsquigarrow (HX^*(X), \smile)$

D.G.A. Graded (cohomology) algebra

$\alpha \smile \beta = 0$ in cohomology

$\smile$ vanishes in cohomology $CX^*(X) \trianglelefteq (C^*(X), \smile)$;

If $\varphi \in CX^*$, $\psi \in C^*$; $\varphi \smile \psi \in CX^*$

$\varphi \in C^*$, $\psi \in CX^*$; $\varphi \smile \psi \in CX^*$

$$
\begin{tikzcd}
 & CX^* \otimes C^* \arrow[dr] & \\
CX^* \otimes CX^* \arrow[ur] \arrow[dr] \arrow[rr, dashed] & & CX^* \\
 & C^* \otimes CX^* \arrow[ur] &
\end{tikzcd}
\qquad CX^* \triangleleft C^*
$$

Let $b \in X$ basepoint; $s: C^* \longrightarrow C^{*-1}$;

$\varphi: X^{n+1} \to R \mapsto (s\varphi)(x_0, \dots, x_{n-1}) = \varphi(b, x_0, \dots, x_{n-1})$

$$CX^k \otimes CX^\ell \longrightarrow CX^{\boxed{k+\ell-1}}$$

$$\varphi * \psi := \boxed{\underbrace{s\varphi}_{k-1} \smile \underbrace{\psi}_{\ell}} + (-1)^k \boxed{\underbrace{\varphi}_{k} \smile \underbrace{s\psi}_{\ell-1}}$$

$HX^k \otimes HX^\ell \xrightarrow{* \neq 0} HX^{k+\ell-1}$ "Roe product in coarse cohomology".

This Roe product in cohomology does depend on the choice of b, for $k, \ell \geq 2$.

- **Thm** (R. 2025): If (X,d) is a metric space, R coeff. ring;

$$\xi : \underset{*}{HX^*(X)} \longrightarrow \underset{*}{HB^*(X)} \quad \text{is a morphism of algebras}$$

- **Corollary** (Computation of the Roe product):
If X is unif. contr. at infinity, then

cup product

$$\left(HX^\bullet(X;R), *\right) \overset{\circledast\, f}{\cong} \left(\operatorname*{colim}_{B\in\mathcal{B}} H^{*-1}(X\setminus B;R), \smile\right)$$

$\circledast$ $\quad f(\alpha * \beta) = (-1)^{|\alpha|} f(\alpha) \smile f(\beta)$

- Brown representability theorem:

$$\text{Homology theories}/\cong \quad \longleftrightarrow \quad \text{Spectra}/\sim$$

Top;
CW-complexes.

$$\text{Coarse homology theories}/\cong \quad \longleftrightarrow \quad \text{Coarse spectra?}/\sim$$

Ulrich Bunke & Alexander Engel
"Homotopy Theory w/ bornological coarse spaces"
Coarse Spectra using ∞-categories.

$$H_{lf} : \mathrm{Top} \to \mathrm{Ab}_{\mathbb{Z}} \quad \underset{\text{limit process}}{\rightsquigarrow} \quad HX_{lf} : \mathrm{Coarse} \to \mathrm{Ab}_{\mathbb{Z}}$$